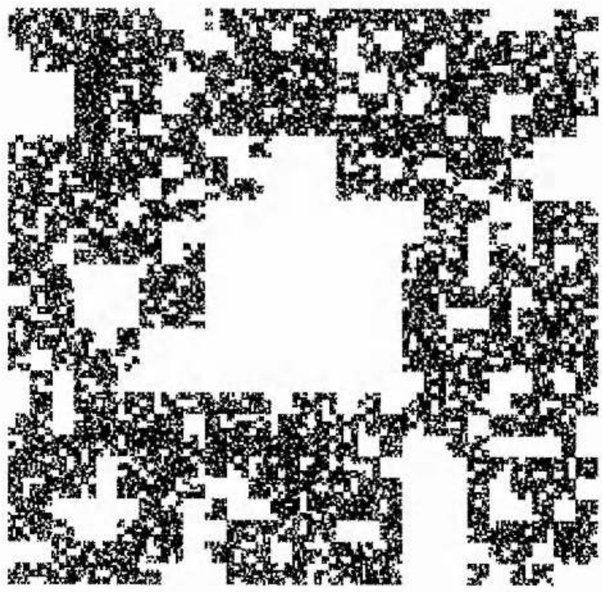
GEOMETRICAL AND TOPOLOGICAL PROPERTIES OF FRACTAL PERCOLATION

Mark E. Orzechowski

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1998

Geometrical and Topological Properties of Fractal Percolation



Mark E. Orzechowski

A Thesis submitted to the
University of St Andrews
for the Degree of
Doctor of Philosophy

September 1997

Th C 410

Abstract

The basic 'fractal percolation' process was first proposed by Mandelbrot in 1974 and takes the following form. Let $M \geqslant 2$ and $p \in [0,1]$; we start with the unit square $C_0 \equiv [0,1]^2$. Divide C_0 into M^2 equal closed squares, each of side-length M^{-1}, in the natural way and retain each of these squares with probability p, or else remove it with probability $1-p$. We let C_1 be the union of those squares retained. The process is now repeated within each square of C_1 to give a new set $C_2 \subseteq C_1$, consisting of squares of side-length M^{-2}. Iterating the construction in the obvious way, we obtain a decreasing sequence of sets $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$ with limit $C_\infty = \bigcap_{n \geqslant 1} C_n$.

The set C_∞ is an example of a random Cantor set, and is typically highly intricate in nature. It may be empty, dust-like or highly connected, depending on the value of p; percolation is said to occur if C_∞ contains large connected components linking opposite sides of the unit square.

In this thesis we shall investigate some of the geometrical and topological properties of C_∞ that hold either almost surely (with probability 1) or with non-zero probability. In particular, the following results are established. We obtain (almost sure) lower and upper bounds on the box-counting dimension of the 'straightest' crossings in C_∞ whenever percolation occurs; we also look at the distribution of the sizes of the connected components and the probability of percolation. In the three-dimensional version of the process, we establish the existence of two distinct phases of percolation, corresponding to the occurrence of paths and surfaces (or 'sheets') in the limit set, and study the limiting behaviour of the phase transition to sheet percolation as $M \to \infty$. We also consider the results of some computer simulations of fractal percolation and present a number of generalisations of the basic process and other closely related constructions.

Declarations

I, Mark E. Orzechowski, hereby certify that this thesis, which is approximately 34,000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

Date ..15/12/97....... Signature of candidate

I was admitted as a research student in October 1994 and as a candidate for the degree of Doctor of Philosophy in October 1995; the higher study for which this is a record was carried out in the University of St Andrews between 1994 and 1997.

Date ...15/12/97....... Signature of candidate

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations for the degree of Doctor of Philosophy in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree.

Date ..15/12/97..... Signature of supervisor ..

Copyright Declaration

Acknowledgements

The financial assistance of EPSRC (grant no. 94000964) for supporting this research is gratefully acknowledged.

I would like to thank all of my colleagues at the University of St Andrews and elsewhere who have helped me over the past three years, most notably the following: Toby O'Neil and John Howroyd, for many interesting lunchtime discussions and diversions; Ben Soares, not only for supplying Figure 5.5a, but for the countless difficulties with the computers he has overcome on my behalf; Andrew Rhodes and Audrey Tan, for locating papers and offering advice; Stephanie van Willigenburg for much sympathy; Ronald Meester, Damien White and the University of Utrecht, for an enjoyable and fruitful recent visit, and the referees of my articles, for several corrections and suggestions.

Special thanks are also due to Jane Alexandra Nickell, for reminding me there is life beyond mathematics, and to my parents for their constant support and encouragement. Finally I would like to thank Professor Kenneth Falconer, without whose expert supervision this thesis could not have been undertaken or completed.

Contents

Overview

This thesis aims to investigate some of the properties of the 'fractal percolation' process and is divided into five main chapters. In Chapter 1, we give two descriptions of the basic process, one intuitive and one rigorous, and present several (well-known) fundamental properties of the limit set, including a proof of the crucial result that the critical probability for percolation is less than 1. We also place this work in context by outlining the historical development of fractal percolation and its applications to other fields.

Chapters 2 and 3 contain important new results, often involving a combination of a detailed geometrical framework and careful probabilistic estimates. In Chapter 2, we calculate almost sure lower and upper bounds on the minimal box-counting dimension of crossings in percolating sets. In Chapter 3, the three-dimensional version of the fractal percolation is considered, along with possibilities for extending the results to still higher dimensions. The main results here are the existence of two distinct phases of percolation as p varies (in one of which there are paths in the retained set, and in the other images of discs) and the limiting behaviour of the critical probability for sheet percolation as the subdivision index M tends to infinity.

Chapter 4 contains a collection of other properties, loosely classed together as 'Numerical Results'. First we examine the distribution of the diameters of the connected components in the super-critical phase of fractal percolation, establishing the expected limiting behaviour as the size decreases. We also calculate an upper bound on the percolation function $\theta(p)$, and present the results of some computer simulations of the process leading to an improved conjectured range of values for the critical probability. Finally in Chapter 5 we consider some other closely related models, including establishing the existence of a unique unbounded cluster in a tiling of the plane by copies of fractal percolation, and finish with a list of open questions and suggestions for further research.

Chapter 1

Introduction

1.1 Description of the fractal percolation process

Consider the following model. We start with the unit square $[0,1]^2$, which we shall denote by C_0, and fix an integer $M \geqslant 2$ and a probability p between 0 and 1. Divide C_0 into M^2 equal closed squares, each of side-length M^{-1}, in the natural way. We now perform independent random coin tosses for each of these squares; with probability p, we retain the square, and with probability $1-p$, we remove the square. Let C_1 be the pointwise union of all the retained squares; the removed squares will take no further part in the construction.

Now we repeat the above process in each of the squares contained in C_1. Divide each such square into M^2 equal closed subsquares, each of side-length M^{-2}, and perform independent coin tosses for each subsquare. Again, each subsquare is retained with probability p, and removed with probability $1-p$; let C_2 be the union of all the retained subsquares. Continuing the process in the obvious way, we obtain a decreasing sequence of closed sets $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$, with a limit set which we shall denote by C_∞. See Figure 1.1 for an illustration of the first three stages of the construction when $M = 3$.

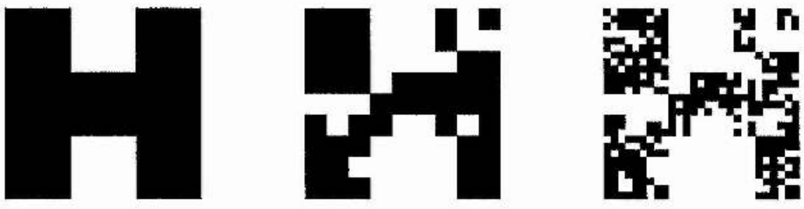


Figure 1.1: Construction of the random Cantor set

The simple process described above has become known as *fractal percolation*, or *Mandelbrot percolation* after Mandelbrot [37]. The limit set C_∞ is an example of a *random Cantor set*, being a random and planar generalisation of the well-known 'middle-third' Cantor set. Typically C_∞ is highly intricate in nature, possessing fine detail and structure when viewed at arbitrarily small scales — in other words, it is a fractal. As we shall see, the limit set typically undergoes radical *phase transitions* as the parameter p varies; it is these discontinuities in behaviour that makes this relatively simple model of great interest to physicists. In this thesis we investigate some of the geometrical and topological properties of C_∞ and other related constructions.

Formal definitions

We now define the fractal percolation process more carefully and introduce some new notation. In practice, however, many of these definitions will often remain in the background, in favour of the more descriptive geometrical version above.

For $\delta > 0$, the *mesh squares of side-length* δ are defined to be the sets of the form $[a\delta, (a+1)\delta] \times [b\delta, (b+1)\delta]$ where $a, b \in \mathbb{Z}$. Fix the *subdivision index* $M \geqslant 2$. For $n \geqslant 0$, the *level-n squares* are the mesh squares of side-length M^{-n} that are contained within the unit square $[0,1]^2$. Let $J = (\mathbb{Z}_M)^2 = \{0, 1, \ldots, M-1\}^2$ and, for $n \geqslant 1$, let $\Sigma_n = J^n$. Elements of Σ_n may be labelled as

$$\mathbf{I} = (\mathbf{i}_1, \ldots, \mathbf{i}_n) = ((i_{1,1}, i_{1,2}), \ldots, (i_{n,1}, i_{n,2}))$$

where $\mathbf{i}_j \in J$ for all $1 \leqslant j \leqslant n$. With each $\mathbf{I} \in \Sigma_n$, we associate the level-n square

$$S[\mathbf{I}] = \left(\sum_{j=1}^{n} i_{j,1} M^{-j}, \sum_{j=1}^{n} i_{j,2} M^{-j} \right) + [0, M^{-n}]^2,$$

so that Σ_n corresponds to the set of all level-n squares. Following this method, observe that if $\mathbf{i}_1, \ldots, \mathbf{i}_{n+1} \in J$, then $S[(\mathbf{i}_1, \ldots, \mathbf{i}_n, \mathbf{i}_{n+1})]$ is a level-$(n+1)$ sub-square of $S[(\mathbf{i}_1, \ldots, \mathbf{i}_n)]$.

Let $\Sigma = \bigcup_{n \geqslant 1} \Sigma_n$ and suppose that we are given a map $\omega \colon \Sigma \to \{0, 1\}$. For each $\mathbf{I} = (\mathbf{i}_1, \ldots, \mathbf{i}_n) \in \Sigma_n$, $n \geqslant 1$, we define the indicator function $1_\omega[\mathbf{I}]$ by

$$1_\omega[\mathbf{I}] = \prod_{j=1}^{n} \omega\left[(\mathbf{i}_1, \ldots, \mathbf{i}_j)\right]$$

so that $1_\omega[\mathbf{I}] = 1$ if and only if $\omega[(\mathbf{i}_1, \ldots, \mathbf{i}_j)] = 1$ for all $1 \leqslant j \leqslant n$. The *level-n pre-fractal* set $C_n = C_n(\omega)$ is then defined by

$$C_n = \bigcup \{S[\mathbf{I}]: 1_\omega[\mathbf{I}] = 1, \mathbf{I} \in \Sigma_n\} \tag{1.1}$$

and the limit set $C_\infty = C_\infty(\omega)$ by

$$C_\infty = \bigcap_{n \geqslant 1} C_n. \tag{1.2}$$

It is not hard to see that for a given map ω, the above construction corresponds to a particular realisation of the fractal percolation process outlined earlier. To turn this construction into a random process, we shall require some definitions from probability theory.

Define the *state space* Ω by $\Omega = \{0,1\}^\Sigma$; then elements ω of Ω may be viewed as maps from Σ into $\{0, 1\}$, and hence correspond to particular realisations of C_∞, or *configurations*. A *cylinder subset* of Ω is given by specifying the values of $\omega(x)$ for x belonging to a certain finite subset of Σ; thus if $X \subseteq \Sigma$ is finite and we fix $w_x \in \{0,1\}$ for each $x \in X$, then

$$A = \{\omega \in \Omega: \omega(x) = w_x \text{ for all } x \in X\} \tag{1.3}$$

represents a cylinder set. We let $\mathcal{F}$ denote the σ-algebra generated by the cylinder subsets of Ω; a *measurable event* is then an element of $\mathcal{F}$, or equivalently an $\mathcal{F}$-measurable subset of Ω.

We denote probability measures on $(\Omega, \mathcal{F})$ by P. Of principal interest to us will be the product probability measure corresponding to fractal percolation with retention probability p, which we shall denote by P_p and define as follows. Fix $0 \leqslant p \leqslant 1$, let X be a finite subset of Σ, let $w_x \in \{0,1\}$ for each $x \in X$ and let A be the cylinder subset given by (1.3). The P_p-measure of A is defined to be

$$P_p(A) = \prod_{x \in X} \big(pw_x + (1-p)(1-w_x)\big). \tag{1.4}$$

By Carathéodory's Extension Theorem, P_p has a unique extension to all of $\mathcal{F}$. It is easy to check that P_p is a probability measure; indeed, when $X = \varnothing$, we have

$$P_p(\Omega) = \prod_{x \in \varnothing} \big(pw_x + (1-p)(1-w_x)\big) = 1. \tag{1.5}$$

This completes the definition of the basic fractal percolation process, as being the family of nested random sets $\{C_n\}_{n\geqslant 1}$ given by (1.1) and with distribution given by the probability measure P_p. We shall examine some of the properties of 'typical' realisations of the limit set C_∞, *i.e.* events that occur P_p-almost surely or with positive P_p-probability. Further notation relating to the process will be introduced later, as and when required.

Phases of the random Cantor set

The properties of a 'typical' realisation of C_∞ alter greatly as p varies between 0 and 1. Consider the two extremes: For very small values of p, we typically remove nearly all of the remaining matter at every stage of the fractal percolation process, so it is very likely that the limit set C_∞ will be empty. On the other hand, for values of p close to 1, very little matter is removed at every stage and so it is plausible that C_∞ will possess a highly connected structure with only small, sparsely distributed holes.

Given a configuration ω, for each $n\geqslant 1$ let $T_n = \operatorname{card}\{\mathbf{I} \in \Sigma_n : 1_\omega[\mathbf{I}] = 1\}$, *i.e.* T_n is the number of level-n squares contained in the pre-fractal set C_n, and let $T_0 = 1$. Since every level-n square of C_n is divided into M^2 subsquares, each of which is retained independently with probability p, we see that $\{T_n\}_{n\geqslant 0}$ forms a Galton–Watson branching process with family size Z, where $Z \sim \operatorname{Bin}(M^2, p)$.

THEOREM 1.1: If $p \leqslant M^{-2}$ then $C_\infty = \varnothing$, P_p-almost surely; if $p > M^{-2}$ then $P_p(C_\infty \neq \varnothing) > 0$.

Proof: The branching process $\{T_n\}_{n\geqslant 0}$ has expected family size $M^2 p$. We use standard results in the theory of branching processes to deduce that if $M^2 p \leqslant 1$ then $\{T_n\}$ becomes extinct almost surely, *i.e.* $C_\infty = \varnothing$, but $\{T_n\}$ survives with non-zero probability if $M^2 p > 1$.

■

The fractal percolation process thus passes through a phase transition at the value $p = M^{-2}$, from $C_\infty = \varnothing$ almost surely to $C_\infty \neq \varnothing$ with positive probability. When C_∞ is non-empty, it is possible for C_∞ either to be 'dust-like', that is, a totally disconnected set of points, or to possess non-trivial connected components larger than a point. These larger components will be the focus of much of our attention; in particular, we shall be interested in components which span the square from left to right.

We label the left-hand edge $\{0\} \times [0,1]$ of the unit square as L and the right-hand edge $\{1\} \times [0,1]$ as R. Fix $M \geqslant 2$ and consider the connected components of the set C_∞, representing fractal percolation performed with subdivision index M. We define *percolation* to occur in C_∞ if C_∞ contains a connected component E such that $E \cap L \neq \varnothing$ and $E \cap R \neq \varnothing$. We shall occasionally consider percolation in other sets, notably the pre-fractal sets C_n, defined in exactly the same way.

Since $\{C_n\}_{n \geqslant 0}$ is a decreasing sequence of compact sets, a connected component of C_∞ can be written as an intersection of a sequence of connected components of C_n, $n \geqslant 0$; hence

$$\{\text{percolation in } C_\infty\} = \bigcap_{n \geqslant 0} \{\text{percolation in } C_n\}. \tag{1.6}$$

We deduce that $\{\text{percolation in } C_\infty\}$ is a measurable event, since each event $\{\text{percolation in } C_n\}$ depends only on the states of finitely many squares, and hence may be written as a finite union of cylinder subsets of Ω.

Define the *percolation function* $\theta(p)$ by

$$\theta(p) = P_p(\text{percolation in } C_\infty)$$

and the *critical probability* p_c by

$$p_c = \inf\{p : \theta(p) > 0\}.$$

It is not at all clear *a priori* that $\theta(p) > 0$ for any value of $p < 1$; this important result will be the subject of Section 1.2. In the meantime we consider the set C_∞ for values of p between M^{-2} and p_c.

THEOREM 1.2: Let $M^{-2} < p < p_c$. Then C_∞ is totally disconnected with probability 1.

Proof: See Chayes *et al.* [8], Theorem 2.

■

Thus we see that C_∞ has a very sudden phase transition at p_c; for $p < p_c$, there is no connected component larger than a point, almost surely, whereas for $p > p_c$, components crossing the whole square from left to right can occur with positive probability. (In fact, the percolation function is discontinuous at p_c, that is, $\theta(p_c) > 0$.)

When $p < p_c$, C_∞ is said to be *sub-critical*; when $p \geqslant p_c$, C_∞ is said to be *super-critical*. Illustrations of typical realisations of C_∞ for the sub- and super-critical phases are shown in Figures 1.2 and 1.3 respectively; the illustration on the cover page shows a realisation for a value of p close to the conjectured value for p_c ($M = 3$, $p = 0.80$).

1.2 Non-triviality of the critical probability

The existence of a non-trivial interval of values of p for which percolation occurs with positive probability is of key importance in the study of random Cantor sets. Several proofs of this result already appear in the literature, but it is worth repeating here in the general case for all $M \geqslant 2$. The first rigorous proof (albeit with slight errors) was given by Chayes *et al.* [8]; their method was corrected in Falconer [21] and streamlined by Dekking and Meester [17]. More recently, Chayes [11] offered an alternative argument based on elementary rescaling properties of C_∞.

We shall follow the method of Chayes *et al.* [8]. Consider the fractal percolation process with $M \geqslant 2$ and $0 \leqslant p \leqslant 1$. For $n \geqslant 0$, we define a level-n square A contained in C_n to be *1-full* if at least $M^2 - 1$ of its M^2 level-$(n+1)$ subsquares are contained in C_{n+1}. Inductively for $m \geqslant 2$, we define A to be *m-full* if at least $M^2 - 1$ of its M^2 subsquares are $(m-1)$-full.

THEOREM 1.3: For $M \geqslant 3$, we have $p_c < 1$.

Proof: First observe that if A_1, A_2 are two neighbouring level-n squares in C_n, each of which is 1-full, then we can find a subsquare of A_1 and a subsquare of A_2 that intersect along an edge, so that $C_{n+1} \cap (A_1 \cup A_2)$ forms a connected unit. Based on this observation, it is easy to see that if the level-0 square $[0,1]^2$ is m-full, then C_m contains a sequence of level-m squares joining L to R, *i.e.*

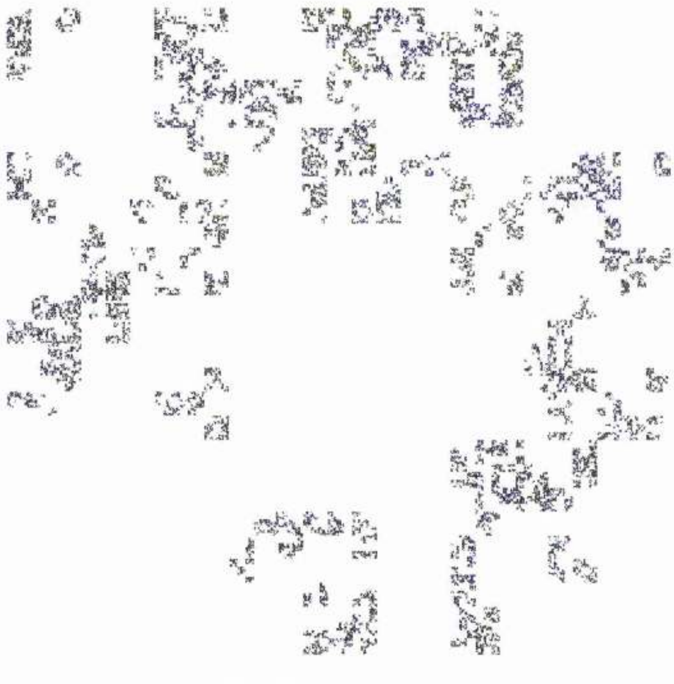


Figure 1.2: Fractal percolation with $M = 3$ and $p = 0.60$

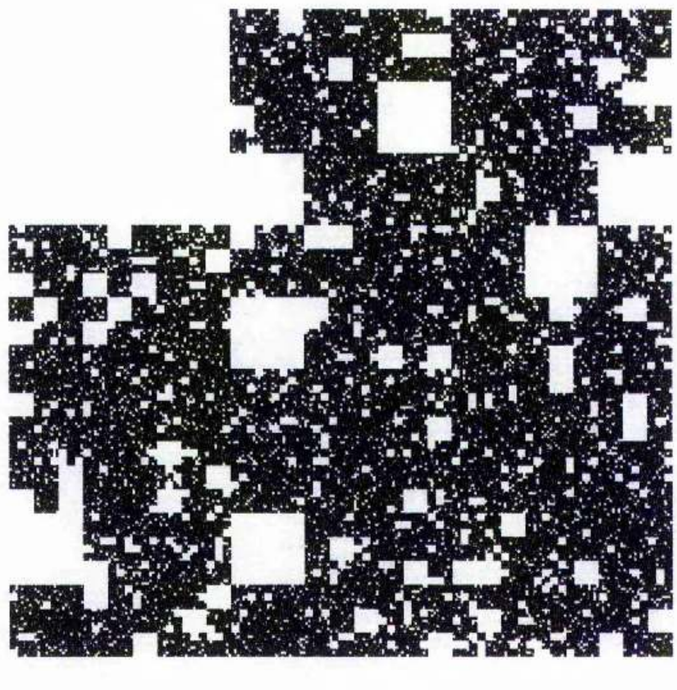


Figure 1.3: Fractal percolation with $M = 3$ and $p = 0.90$

percolation occurs in C_m.

Let $\phi_m = P_p([0,1]^2 \text{ is } m\text{-full})$. By (1.6), we have

$$\theta(p) = P_p\Big(\bigcap_{n\geqslant 0}\{\text{percolation in } C_n\}\Big) = \lim_{n\to\infty} P_p(\text{percolation in } C_n) \qquad (1.7)$$

and hence

$$\theta(p) \geqslant \lim_{m\to\infty} \phi_m. \qquad (1.8)$$

Thus it remains to show that for some value of $p < 1$, we have $\lim_{m\to\infty} \phi_m > 0$.

Now $\phi_1 = p^{M^2} + M^2 p^{M^2-1}(1-p)$ and for $m \geqslant 1$, we have

$$\begin{aligned}\phi_{m+1} &= p^{M^2}\phi_m^{M^2} + p^{M^2}M^2\phi_m^{M^2-1}(1-\phi_m) + M^2p^{M^2-1}(1-p)\phi_m^{M^2-1} \\ &= M^2p^{M^2-1}\phi_m^{M^2-1} - (M^2-1)p^{M^2}\phi_m^{M^2}. \qquad (1.9)\end{aligned}$$

Defining $f_p\colon [0,1] \to [0,1]$ by

$$f_p(x) = M^2p^{M^2-1}x^{M^2-1} - (M^2-1)p^{M^2}x^{M^2},$$

we thus have a dynamical system $\{\phi_m\}_{m\geqslant 0}$ given by $\phi_0 = 1$ and $\phi_{m+1} = f_p(\phi_m)$ for $m \geqslant 0$. Differentiating f_p, we obtain

$$\begin{aligned}f_p'(x) &= M^2(M^2-1)p^{M^2-1}x^{M^2-2} - M^2(M^2-1)p^{M^2}x^{M^2-1} \\ &= M^2(M^2-1)p^{M^2-1}x^{M^2-2}(1-px) > 0 \qquad (1.10)\end{aligned}$$

and hence we observe that f_p is an increasing function with $f_p(0) = 0$ and $f_p(1) < 1$. It turns out that f_p has a fixed point in the interval $[1-M^4(1-p)^2, 1)$ whenever $p \geqslant 1 - M^{-5}/3$; see the proof of Lemma 2.14 for more details of this result in a more general setting.

We conclude that if $p \geqslant 1 - M^{-5}/3$ then $\phi_m \geqslant 1 - M^4(1-p)^2$ for all $m \geqslant 1$; in particular, we have $\lim_{m\to\infty}\phi_m \geqslant 1 - M^4(1-p)^2 > 0$, implying that $p_c \leqslant 1 - M^{-5}/3 < 1$, as required.

∎

Of course, the method of Theorem 1.3 is extremely inefficient for producing bounds on p_c; at best, when $M = 3$, it gives $\theta(p) \geqslant 0.999\,84$ for $p \geqslant 0.998\,63$. Some improvement can be made by observing that we can weaken the definition of a 1-full square; in the case $M = 3$, certain combinations of only seven out of the nine subsquares are sufficient to ensure that neighbouring 1-full squares form

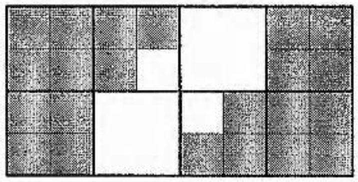


Figure 1.4: $C_{n+2} \cap (A_1 \cup A_2)$ is disconnected

a connected unit. Xu and Su [59] have performed these calculations, reporting an upper bound on p_c of 0.983 03. However, substantial improvements based on these techniques appear unlikely.

So far, we have only considered $M \geqslant 3$. When $M = 2$, the method of Theorem 1.3 cannot be used directly, since it is possible for the level-$(n+2)$ subsquares of two neighbouring level-n 2-full squares A_1, A_2 to be disconnected, as shown in Figure 1.4. However, as observed by Chayes *et al.* [8] and Dekking and Meester [17], we can use a coupling argument to compare the cases $M = 2$ and $M = 4$. We let $\theta(p; M)$ denote the usual percolation function for fractal percolation with index parameter M and retention probability p, and let $p_c(M) = \inf\{p: \theta(p;M) > 0\}$.

PROPOSITION 1.4: Suppose that p, q satisfy $1 - p^{M^2+1} \leqslant (1-q)^{M^2}$. Then we have $\theta(p; M) \geqslant \theta(q; M^2)$.

Proof: We compare the level-2 set $C_2[M]$ for $M \times M$ fractal percolation with probability p and the level-1 set $C_1[M^2]$ for $M^2 \times M^2$ fractal percolation with probability q. Let A be a fixed mesh square of side-length M^{-1} and let $B_1, \ldots, B_{M^2}$ denote the mesh squares of side-length M^{-2} contained in A.

In the first model, we have

$$\begin{aligned} P_p\big(A \subseteq C_2[M]\big) &= P_p\big(A \subseteq C_1[M]\big) P_p\big(\forall i,\, B_i \subseteq C_2[M] \,\big|\, A \subseteq C_1[M]\big) \\ &= p.p^{M^2} = p^{M^2+1}. \end{aligned} \tag{1.11}$$

In the second model, since each of the B_i, $1 \leqslant i \leqslant M^2$, is retained independently with probability q, we have

$$P_q\big(\mathrm{int}(A \cap C_1[M^2]) = \varnothing\big) = (1-q)^{M^2}. \tag{1.12}$$

Therefore when the hypothesis of the proposition is satisfied, we have

$$P_p\big(A \not\subseteq C_2[M]\big) \leqslant P_q\big(\operatorname{int}(A \cap C_1[M^2]) = \varnothing\big). \tag{1.13}$$

In other words, the probability that some subsquare of A has been removed in $C_2[M]$ is no greater than the probability that *all* subsquares of A have been removed in $C_1[M^2]$. Since this inequality is repeated across $[0,1]^2$, we deduce that

$$P_p\big(\text{percolation in } C_2[M]\big) \geqslant P_q\big(\text{percolation in } C_1[M^2]\big). \tag{1.14}$$

A corresponding version of inequality (1.13) applies equally to mesh squares A of side-length $M^{-(2n-1)}$ for all $n \geqslant 1$; thus we can similarly compare the sets $C_{2n}[M]$ and $C_n[M^2]$ to deduce that

$$P_p\big(\text{percolation in } C_{2n}[M]\big) \geqslant P_q\big(\text{percolation in } C_n[M^2]\big). \tag{1.15}$$

Taking limits as $n \to \infty$, we conclude that $\theta(p; M) \geqslant \theta(q; M^2)$, as required.

■

COROLLARY 1.5: $p_c(2) < 1$.

Proof: Let $M = 2$, $q > p_c(4)$ and let p satisfy

$$\big(1 - (1-q)^{M^2}\big)^{1/(M^2+1)} \leqslant p < 1.$$

By Theorem 1.4, we have $\theta(p; 2) \geqslant \theta(q; 4) > 0$, and hence $p_c(2) \leqslant p < 1$.

■

1.3 Some basic results

In this section we collect together some important definitions and results that will be useful later in our study of fractal percolation.

Increasing events and the FKG inequality

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and denote points of Ω by ω. Let A be an event, that is, an $\mathcal{F}$-measurable subset of Ω, and let $1_A : \Omega \to \{0, 1\}$ be the indicator function of A, taking the value 1 if $\omega \in A$ and 0 otherwise.

Suppose that we have a partial order '$\leqslant$' on the configurations $\omega \in \Omega$. (When Ω is the state space corresponding to fractal percolation (or discrete percolation), there is a natural partial order on Ω: We say that $\omega_1 \leqslant \omega_2$ if and only if $\omega_1[\mathbf{I}] \leqslant \omega_2[\mathbf{I}]$ for all $\mathbf{I} \in \Sigma$, so that every square (or site, or bond) selected in ω_1 is also selected in ω_2.) We define the event A to be *increasing* if $1_A(\omega_1) \leqslant 1_A(\omega_2)$ for all $\omega_1, \omega_2 \in \Omega$ such that $\omega_1 \leqslant \omega_2$. In addition, we define A to be *decreasing* if its complement A^c is increasing.

Many of the events that we shall consider in the context of the fractal percolation process are increasing. In particular, the events {percolation in C_n} and {percolation in C_∞} are increasing; to see this, simply observe that if ω_1 is a configuration in which percolation occurs and $\omega_1 \leqslant \omega_2$, then ω_2 is obtained from ω_1 simply by adding extra squares (and not removing any), so that percolation also occurs in ω_2.

The usefulness of increasing events comes from the following important correlation inequality.

THEOREM 1.6: (FKG inequality)
Let $(\Omega, \mathcal{F}, P)$ be a probability space and let A and B be two increasing events (or two decreasing events) that depend on at most countably many random variables. Then

$$P(A \cap B) \geqslant P(A)P(B).$$

Theorem 1.6 was first proved by Harris [30] in the case of discrete percolation on the lattice with the natural product probability measure; its scope was widened by Fortuin, Kasteleyn and Ginibre [25], after whom it is named.

Pasting results

The FKG inequality gives us many useful intersection results. For example, let A denote the event {percolation from left to right}, let B denote the event {percolation from top to bottom} and observe that both these events are increasing. Whenever $p \geqslant p_c$, we have $P_p(A \cap B) \geqslant P_p(A)P_p(B) > 0$ by Theorem 1.6, and hence we deduce that the critical probability for the event $A \cap B$ equals p_c.

We can also use the FKG inequality to obtain a lower bound on the probability of more complicated events. For $J, K \in \mathbb{N}$, let $C_\infty^{J,K}$ denote the set formed by placing JK independent copies of fractal percolation side by side in the rectangle $[0, J] \times [0, K]$. Let

$$\theta^{J,K}(p) = P_p(\text{percolation from left to right in } C_\infty^{J,K})$$

and let $p_c^{J,K} = \inf\{p: \theta^{J,K}(p) > 0\}$.

LEMMA 1.7: If $\theta^{1,2}(p) > 0$ then $\theta^{1,1}(p) > 0$.

Proof: See Dekking and Meester [17], Lemma 5.1. Essentially, the lemma is proved by taking a left to right crossing of $[0, 1] \times [0, 2]$ and reflecting and translating this crossing several times. These copies are then pasted together to build up a crossing of $[0, M] \times [0, 2]$; the FKG inequality tells us that such a crossing exists with positive probability. Finally the whole set is rescaled by a factor of $1/M$ to provide a crossing of $[0, 1] \times [0, 1]$.

■

Note that the same techniques used in the proof of Lemma 1.7 can be used to build up crossings of longer rectangles, simply by reflecting and translating as many times as required. Also observe that $\theta^{1,1}(p) > 0$ implies $\theta^{1,2}(p) > 0$ trivially; hence we have the following useful result.

LEMMA 1.8: (Pasting lemma)
If $\theta^{1,1}(p) > 0$ then $\theta^{J,K}(p) > 0$ for all $J, K \geqslant 1$.

COROLLARY 1.9: For all $J, K \geqslant 1$, we have $p_c^{J,K} = p_c$.

Box-counting dimension and Hausdorff dimension

The *lower box dimension* $\underline{\dim}_B(E)$ of a non-empty bounded set $E \subseteq \mathbb{R}^n$ is given by

$$\underline{\dim}_B(E) = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta} \tag{1.16}$$

and the *upper box dimension* $\overline{\dim}_B(E)$ by

$$\overline{\dim}_B(E) = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta} \tag{1.17}$$

where $N_\delta(E)$ is the smallest number of cubes of side-length δ that cover E. If $\underline{\dim}_B(E) = \overline{\dim}_B(E)$, we define the *box-counting dimension* $\dim_B(E)$ of E to be the common value. Observe that we may replace the limit $\delta \to 0$ in (1.16) and (1.17) by the limit $n \to \infty$ on substituting $\delta = M^{-n}$, and also that the dimensions are unaltered if we consider only mesh cubes of side-length M^{-n} covering E (see Section 3.1 of Falconer [21]).

The following proposition is a standard result from the theory of branching processes.

PROPOSITION 1.10: Let $\{T_n\}_{n\geqslant 0}$ be a branching process where the family size has expected value μ and finite variance. Then with probability 1, there exists a constant $0 \leqslant c < \infty$ such that $T_n \sim c\mu^n$ as $n \to \infty$; moreover, if $T_n \not\to 0$ as $n \to \infty$, then $c > 0$ almost surely.

Proof: See Harris [31], Chapter I, Section 8.1.

■

We use Proposition 1.10 to obtain an almost sure value for the box-counting dimension of the set C_∞ in the plane.

THEOREM 1.11: Conditional on $C_\infty \neq \varnothing$, then with probability 1 we have $\dim_B(C_\infty) = 2 + \log p / \log M$.

Proof: Consider the sequence $\{T_n\}_{n\geqslant 0}$, where T_n denotes the number of level-n squares contained in the set C_n, and recall that this sequence forms a branching process with family size $Z \sim \mathrm{Bin}(M^2, p)$. Observe that $\mathbf{E}(Z) = M^2 p$ (where $\mathbf{E}$ denotes expectation with respect to P_p) and $\mathrm{Var}(Z) = M^2 p(1-p) < \infty$, and hence we deduce from Proposition 1.10 that with probability 1, there exists $c > 0$ such that $T_n \sim c(M^2 p)^n$ as $n \to \infty$. Since $N_{M^{-n}}(C_\infty) \leqslant T_n$ for all $n \geqslant 0$, we have

$$\overline{\dim}_B(C_\infty) \leqslant \limsup_{n\to\infty} \frac{\log T_n}{\log M^n} = \frac{\log M^2 p}{\log M} = 2 + \frac{\log p}{\log M}. \tag{1.18}$$

almost surely.

The lower bound is slightly more difficult, since some of the level-n squares contained in C_n may have empty intersection with C_∞. Let $f\colon [0,1] \to [0,1]$

denote the probability generating function for the family size in the branching process $\{T_n\}$; the expected family size is then equal to $f'(1)$ and the extinction probability q is given by the smallest solution of $f(s) = s$. Let $n \geqslant 0$ and let S be a level-n square contained in C_n; say that S *survives* if there exists an infinite line of descendents from S.

We create a new process $\{\tilde{T}_n\}_{n\geqslant 0}$ from $\{T_n\}_{n\geqslant 0}$ by defining $\tilde{T}_n$ to be the number of level-n squares S contained in C_n that survive. It follows from Theorem 1, Section I.12 of Athreya and Ney [2] that $\{\tilde{T}_n\}$ is equivalent in distribution to a branching process with family size given by the random variable X having generating function

$$\tilde{f}(s) = \frac{1}{1-q}\Big(f\big((1-q)s+q\big) - q\Big). \tag{1.19}$$

Therefore

$$\mathbf{E}(X) = \tilde{f}'(1) = (1-q)^{-1}(1-q)f'\big((1-q)1+q\big) = f'(1) = M^2 p. \tag{1.20}$$

By Proposition 1.10, if $C_\infty \neq \varnothing$ then with probability 1 there exists $0 < c < \infty$ such that $\tilde{T}_n \sim c(M^2p)^n$, and so

$$\underline{\dim}_B(C_\infty) \geqslant \liminf_{n\to\infty} \frac{\log \tilde{T}_n}{\log M^n} = \frac{\log M^2 p}{\log M} = 2 + \frac{\log p}{\log M}. \tag{1.21}$$

Combining (1.18) and (1.21), we see that $\dim_B(C_\infty) = 2 + \log p/\log M$ almost surely, conditional on $C_\infty \neq \varnothing$.

∎

From the point of view of ease of calculation, box-counting dimension is the most natural definition of fractal dimension for us to use. The Hausdorff dimension, $\dim_H(C_\infty)$, of C_∞ is however also important; see (amongst many others) Falconer [21] or Rogers [51] for a definition and discussion of Hausdorff measure and dimension. Several authors, including Hawkes [32] and Falconer [20], have tackled the problem of finding the Hausdorff dimension of various classes of random objects, including random Cantor sets, from different angles. They all show that

$$\dim_H(C_\infty) = 2 + \log p/\log M \tag{1.22}$$

almost surely, conditional on $C_\infty \neq \varnothing$. More generally, in the case of the d-dimensional random Cantor set to be introduced in Section 3.1, we have

$$\dim_H(C_\infty) = \dim_B(C_\infty) = d + \log p/\log M \tag{1.23}$$

almost surely, conditional on $C_\infty \neq \varnothing$.

1.4 Historical background and applications

Early models

The origins of fractal percolation can be traced as far back as the Russian paper of Novikov and Stewart [46] in 1964. The process they describe and the notation they use differ markedly from those favoured more recently; nevertheless, their model contains several of the same key features as the one which was later to evolve from it.

The essential difference between the Novikov–Stewart model and our random Cantor sets lies in the subdivision and selection steps. Whereas we retain each of the M^2 subsquares independently at random with probability p, the method used by Novikov and Stewart introduces more constraints. They fix an integer N, $0 \leqslant N \leqslant M^2$; then for all $n \geqslant 0$, every level-n retained square is divided into M^2 equal subsquares of which *exactly* N are retained. (The subsquares to be retained are chosen at random according to some uniform distribution.)

The result of iterating this process *ad infinitum* is a random fractal which has box-counting dimension $\log N / \log M$ (surely, not almost surely). This set displays many of the same properties as C_∞; for $N < M$, the set is totally disconnected, whilst for $N > M^2 - M/2$, it is easy to show that percolation occurs. We are therefore able to define the percolation function and phase transition and estimate the critical value for N in exactly the same way as for C_∞. However, the removal of independence in this model serves to make any probabilistic calculations much more difficult; in addition, because N can take only finitely many values, any variations in behaviour will be of a discrete nature.

Novikov and Stewart introduced their model in the context of a problem concerning turbulent flows. This was also the motivation for Mandelbrot [37] in 1974, when fractal percolation in its present form was first postulated under the title of 'canonical curdling'. Mandelbrot developed his ideas further in [38], although his arguments are intuitive and centre around the 'critical dimensions'. Further conjectures were made by Mandelbrot [39] in 1983; it was not however until 1988 that the model first received rigorous mathematical attention.

Recent results

Chayes, Chayes and Durrett [8] established several fundamental properties of random Cantor sets which underpin all subsequent developments. They proved the existence of three basic phases (empty, totally disconnected and percolating) and claimed that the phase transition to percolation is discontinuous, *i.e.* $\theta(p_c) > 0$. However, their proof of the latter result is not quite accurate; the event G used in the FKG inequality on page 318 is not in fact increasing.

Dekking and Meester [17] built on the work of Chayes *et al.* using the terminology of random substitutions. They extended the classification of random Cantor sets to a total of six phases by looking at the projection of C_∞ onto one of the axes, and proved that at least one of the phases (phase III) is in fact omitted. In addition, Dekking and Meester completed the proof of the discontinuity of the phase transition at p_c and introduced the random Sierpiński carpet, to which we shall return in Section 5.2.

Chayes and Chayes [7] considered the large M limit of the fractal percolation process in the plane. They proved that

$$p_c(M) \to p_c(\mathbb{Z}^2) \quad \text{as } M \to \infty \tag{1.24}$$

where $p_c(\mathbb{Z}^2)$ denotes the critical probability for site percolation in the discrete square lattice. Falconer and Grimmett [23, 24] extended this result to higher dimensional versions of fractal percolation in a somewhat unexpected way; see Section 3.1 for further details. Chayes *et al.* [9] concentrated exclusively on fractal percolation in three dimensions; again, this is covered in Chapter 3.

In addition, Falconer and Grimmett [24] completed the Dekking–Meester classification of random Cantor sets in two dimensions, by showing that the process passes directly from phase II to phase V at $p = M^{-1}$, omitting phases III and IV. Unfortunately this result makes the later (independent) work of Wu and Liu [58] largely redundant, although the lower bound they claim on the critical probability, $p_c(3) > 0.6346$, is still of interest.

The 1992 paper of Meester [41] introduced an alternate definition of percolation by arcwise-connected components. Given a realisation C_∞ of fractal percolation, we define *arc-percolation* to occur in C_∞ if there exists a continuous map $\Gamma\colon [0,1] \to C_\infty$ such that $\Gamma(0) \in \{0\} \times [0,1]$, $\Gamma(1) \in \{1\} \times [0,1]$ and

$\Gamma(s_1) \neq \Gamma(s_2)$ whenever $s_1 \neq s_2$. Clearly {arc-percolation} $\subseteq$ {percolation}; moreover, Meester proved that, at least in two dimensions, the two notions are probabilistically equivalent, *i.e.*

$$P_p(\text{arc-percolation}) = P_p(\text{percolation}) = \theta(p). \tag{1.25}$$

This result will be useful for a number of our geometrical lemmas, since it enables us to use the two definitions of percolation interchangeably (up to sets of measure zero). However, it is an open problem to extend (1.25) to higher dimensions; it is not even clear whether arc-percolation is then a measurable event or not.

The work of Chayes [12, 13] on (the absence of) directed fractal percolation and the length of the crossings in the super-critical phase will be considered in Chapter 2. Other important references include Dekking and Grimmett [16], Falconer [20], Graf [28] and the survey paper of Chayes [11].

Applications

As mentioned above, the study of fractal percolation grew out of a problem in turbulent fluid flow; indeed, the language used by Mandelbrot in his description of the process talks of 'eddies' and 'subeddies'. Even before the paper of Novikov and Stewart, however, random Cantor sets of a sort had been used by Fournier d'Albe [26] and Hoyle [33] to model the distribution of matter in the universe by a cascade process. According to Hoyle, a galactic cloud of radius R_0 contracts by a factor of $k^{2/3}$ and then divides into k equal fragments, each of radius R_0/k. This process is then iterated in the obvious way so that the limiting set is fractal in nature; however, no consideration is given to the spatial distribution of the fragments.

As is suggested by its name, 'percolation' relates to a physical phenomenon. The traditional model of bond percolation on a discrete lattice can be used to represent the movement of fluid in a large porous stone; open bonds correspond to passageways through which fluid can travel. On a similar theme, fractal percolation can be used to model the hydraulic properties of a sample of soil in which pores occur at random locations and across a large range of spatial scales. For $n \geqslant 1$, the retained squares of side-length M^{-n} in C_n correspond to occupied cells, and these are given an impedance of R_n; the vacant squares correspond to unoccupied cells, and these are given an impedance of r_n. Crawford [14] investigates the conductivity of such a model and shows the existence of a phase transition in p, between the conductivity essentially obeying a power law dependency on the length scale of the measurement, and being dominated by the largest connected pores in the structure.

Su and Yan [54] have translated the terminology of fractal percolation to model 'ductile fracture' in plate specimens. The removed squares correspond to 'microvoids' in the specimen; if the probability for forming microvoids is sufficiently high, then ductile fracture occurs with probability 1. Dekking [15] attempted to use a modified model of fractal percolation, with interaction between neighbouring squares, to represent sections of a human lung, but had limited success. Mandelbrot aerogels were introduced by Machta [35] and will be discussed in Section 5.2.

Brownian motion

The application of perhaps the most direct interest to mathematicians concerns a recent important paper of Peres [49] linking Brownian motion paths in $\mathbb{R}^d$ to fractal percolation at an appropriate value of p. Let $C_\infty(p)$ denote d-dimensional fractal percolation with subdivision index $M = 2$ and retention probability p (see Section 3.1 for a definition). Two random (Borel) sets A and B in $\mathbb{R}^d$ are defined to be *intersection-equivalent* in an open set U if there exist constants $0 < c_1 \leqslant c_2 < \infty$ such that for every closed set $\Lambda \subset U$, we have

$$c_1 \leqslant \frac{P(A \cap \Lambda \neq \varnothing)}{P(B \cap \Lambda \neq \varnothing)} \leqslant c_2. \tag{1.26}$$

Using results from the classical percolation theory for Brownian motion and capacities on trees and in Euclidean space, Peres proves the following result.

THEOREM 1.12:

(*i*) For $d \geqslant 3$, the range of d-dimensional Brownian motion is intersection-equivalent to $C_\infty(2^{2-d})$ in $(0,1)^d$.

(*ii*) For $d = 2$, any Borel set A such that $P_p(C_\infty(p) \cap A \neq \varnothing) > 0$ for some $p < 1$ has non-empty intersection with the range of 2-dimensional Brownian motion, almost surely.

(Peres also gives analogous versions for stable symmetric processes and for random walks on the lattices $\mathbb{Z}^d$.) With this powerful result at hand, it becomes easy to derive the following famous probabilistic statements of Dvoretzky, Erdős *et al.* [18, 19] on the intersections of independent Brownian paths.

COROLLARY 1.13:

(i) For $d \geqslant 4$, two independent Brownian paths in $\mathbb{R}^d$ started at different points are almost surely disjoint.

(ii) In $\mathbb{R}^3$, any two paths intersect almost surely, but three paths have no point of mutual intersection, almost surely.

(iii) In $\mathbb{R}^2$, any finite number of paths have points of mutual intersection, almost surely.

Proof: To prove *(i)*, it suffices to show that two independent Brownian paths $B(t), B'(t)$ in $\mathbb{R}^4$ have no point of intersection in the unit cube, almost surely. By Theorem 1.12, $\{B(t)\}_{t\geqslant 0}$ and $\{B'(t)\}_{t\geqslant 0}$ are intersection-equivalent to $C_\infty(1/4)$ and $C'_\infty(1/4)$ in $(0,1)^d$. It is then easy to show that $\{B(t)\}\cap\{B'(t)\}$ is intersection-equivalent to $C_\infty(1/16)$; the result follows since $C_\infty(1/16) = \varnothing$ with probability 1 by Theorem 1.1. The proofs of *(ii)* and *(iii)* are similar; corresponding statements for symmetric stable processes and random walks also hold.

■

Chapter 2

Box-Counting Dimension of Crossings

In this chapter we shall address the problem of finding the fractal dimension of crossings of the random Cantor set C_∞ in the plane. By a *percolating path* or *crossing* we mean a continuous path $\Gamma\colon [0,1] \to C_\infty$ such that $\Gamma(0) \in L = \{0\} \times [0,1]$ and $\Gamma(1) \in R = \{1\} \times [0,1]$. By (1.25), crossings occur in C_∞ with non-zero probability when $p \geqslant p_c$, where p_c is the critical probability for percolation.

From a physical viewpoint, estimates on the dimension of crossings are important for assessing the speed with which a fluid permeates through a medium — the higher the dimension, the more tortuous the route the fluid has to travel. The ideal result in this direction would be to prove the existence of, and then to calculate, a sharp, almost sure (conditional on percolation) value for the dimension of the 'straightest' paths (*i.e.* those with least dimension) in any realisation of C_∞.

Which definition of dimension we choose is a matter of preference and ease of use; a case can be made for each of Hausdorff, box-counting and divider dimensions (see Falconer [21] for a detailed discussion). In practice, as is often the case, it easier to work with box-counting dimension than with Hausdorff dimension; even then, the existence of a sharp value for the minimal dimension of crossings seems to be a difficult problem, and so we present almost sure bounds on the lower and upper box dimensions as functions of p.

The only previous investigation into the dimension of crossings is that of Chayes [13]. In an early preprint, he claimed that an argument similar to

Theorem 1.3 shows the existence of a phase in which rectifiable crossings of finite length can exist with positive probability. This is, however, corrected in later versions; he proves that with probability 1, all percolating paths are non-rectifiable and actually have a lower box dimension of at least $1+\zeta$ for some $\zeta > 0$. The arguments involved in the proof are intricate, involving showing that directed percolation is a necessary and sufficient condition for $\zeta = 0$. Here *directed percolation* is said to occur if there exist percolating paths $\Gamma(s)$ which move only to the right as s increases, without 'doubling back'; Chayes [12] shows that directed percolation occurs with probability zero for $p < 1$. No value for ζ is calculated, nor is any dependence of ζ on p exhibited.

2.1 Definitions and results

Recall the following definitions of dimension from Section 1.3: The *lower box dimension* $\underline{\dim}_B(E)$ of a non-empty subset E of $[0,1]^2$ is given by

$$\underline{\dim}_B(E) = \liminf_{n\to\infty} \frac{\log N^{(n)}(E)}{\log M^n} \tag{2.1}$$

and the *upper box dimension* $\overline{\dim}_B(E)$ by

$$\overline{\dim}_B(E) = \limsup_{n\to\infty} \frac{\log N^{(n)}(E)}{\log M^n} \tag{2.2}$$

where $N^{(n)}(E)$ is the number of level-n squares that intersect E.

If $\gamma\colon [0,1] \to [0,1]^2$ is a continuous path, we shall sometimes identify γ with $\mathrm{Im}(\gamma) = \gamma([0,1])$ by defining $N^{(n)}(\gamma) = N^{(n)}(\gamma([0,1]))$, $\underline{\dim}_B(\gamma) = \underline{\dim}_B(\gamma[0,1])$ and $\overline{\dim}_B(\gamma) = \overline{\dim}_B(\gamma[0,1])$. The path γ is said to satisfy a *Hölder condition of exponent* α if there exists a constant $c > 0$ such that

$$|\gamma(t_1) - \gamma(t_2)| \leqslant c|t_1 - t_2|^\alpha \tag{2.3}$$

for all $t_1, t_2 \in [0,1]$; note that in this definition we necessarily have $\alpha \leqslant 1$. It is easily shown that any path γ satisfying a Hölder condition of exponent α has dimension $\underline{\dim}_B(\gamma) \leqslant 1/\alpha$; to see this, observe that if $\{U_i\}$ is a cover of $[0,1]$ by intervals of length ε, then $\{\gamma(U_i)\}$ is a cover of $\gamma([0,1])$ by sets of diameter at most $\delta = c\varepsilon^\alpha$; since $N_\varepsilon([0,1]) \leqslant 2/\varepsilon$, we have $N_\delta(\gamma) \leqslant 2c^{1/\alpha}\delta^{-1/\alpha}$.

It will sometimes be convenient to formulate the fractal percolation process in the following alternative way: For $n \geqslant 1$, we let C'_n be the union of mesh squares of side-length M^{-n} contained in $[0,1]^2$, where each square is retained at random with probability p independently of all other squares. It is easily seen that the intersection of these sets, $C_\infty = \bigcap_{n\geqslant 1} C'_n$, defines an identically equivalent process to the usual random Cantor set as described in Section 1.1; however in this way, we have removed from the definition of the level-n set any dependence on the previous levels. In addition, we define $C_{n_1}^{n_2} = \bigcap_{n=n_1}^{n_2} C'_n$.

THEOREM 2.1: There exists a constant $v = v(M) > 0$ such that with probability 1, every percolating path $\Gamma\colon [0,1] \to C_\infty$ satisfies

$$\underline{\dim}_B(\Gamma) \geqslant 1 + v(1-p)^4 |\log(1-p)|^{-3}. \tag{2.4}$$

Theorem 2.1 extends the result of Chayes [13]. Note that Theorem 2.1 implies a (very small) lower bound for p_c, since no path can have dimension greater than 2, but this will not improve on previous bounds.

COROLLARY 2.2: With probability 1, no percolating path Γ satisfies a Hölder condition of exponent α for any $\alpha > \left(1 + v(1-p)^4 |\log(1-p)|^{-3}\right)^{-1}$.

THEOREM 2.3: Consider the set $C_\infty^{J,K}$ (where $J, K \in \mathbb{N}$) consisting of JK independent copies of C_∞ placed together in the rectangle $[0,J] \times [0,K]$. Then with probability 1, every path $\Gamma\colon [0,1] \to C_\infty^{J,K}$ crossing $[0,J] \times [0,K]$ from left to right satisfies

$$\underline{\dim}_B(\Gamma) \geqslant 1 + v(1-p)^4 |\log(1-p)|^{-3} \tag{2.5}$$

where v is the same constant as in Theorem 2.1.

Proof: Theorem 2.3 is proved in a similar manner to Theorem 2.1. All the arguments in Section 2.2 for the square $[0,1]^2$ will adapt to the rectangle $[0,J] \times [0,K]$. (The proof does not quite follow from Theorem 2.1 by rescaling alone since it is just conceivable that there exists a path of smaller dimension crossing $[0,J] \times [0,K]$ which does not cross any smaller square from left to right.)

■

COROLLARY 2.4: Almost surely, there exists no non-constant path $\Gamma\colon [0,1] \to C_\infty$ such that

$$\underline{\dim}_B(\Gamma) < 1 + v(1-p)^4 |\log(1-p)|^{-3},$$

where v is the same constant as in Theorem 2.1.

Proof: Observe that every non-constant path crosses a rectangle of the form $[jM^{-n}, (j+1)M^{-n}] \times [0,1]$ or $[0,1] \times [jM^{-n}, (j+1)M^{-n}]$ for some $n \geqslant 0$ and $0 \leqslant j \leqslant M^n - 1$. By Theorem 2.3 applied to this rectangle rescaled by a factor of M^n, with probability 1 every path Γ crossing the rectangle satisfies (2.5). Since there are only countably many such rectangles, we deduce that

$$P_p\big(\exists \text{ non-constant path } \Gamma \text{ s. t. } \underline{\dim}_B(\Gamma) < 1 + v(1-p)^4 |\log(1-p)|^{-3}\big) = 0. \tag{2.6}$$

■

Thus we have a lower bound on the lower box dimension of all non-trivial paths $\gamma\colon [0,1] \to C_\infty$. Of course, this lower bound only has a meaningful interpretation at and above the critical point p_c, below which there exist no connected components larger than a point, almost surely.

As a partial converse, in Section 2.3 we shall obtain an upper bound on the upper box dimension of crossings for values of p close to 1. This upper bound takes the form of an almost sure (conditional on percolation) bound on the minimal value of the upper box dimension of all crossings contained in C_∞.

THEOREM 2.5: Suppose that $M \geqslant 3$ and $1 - M^{-5}/15 \leqslant p < 1$. Then there exists a constant $u = u(M) > 0$ such that

$$P_p\big(\exists \text{ crossing } \Gamma \text{ s.t. } \overline{\dim}_B(\Gamma) \leqslant \beta \,\big|\, \text{percolation}\big) = 1 \tag{2.7}$$

where

$$\beta = \beta(M,p) = 1 + \frac{\log 3}{\big|\big(\log(1-p)\big)/5 - u\big|}.$$

(In fact, we can take $u = \log\big((1 - M^{-2})^{-1/2} 15^{1/5} M\big)$ in Theorem 2.5). Strictly speaking, the measure P_p in (2.7) is actually the completion of the usual probability measure P_p. We shall prove Theorem 2.5 by constructing a Borel-measurable event Λ such that $P_p(\Lambda) = 1$ and

$$\Lambda \cap \{\text{percolation}\} \subseteq \{\exists \text{ crossing } \Gamma \text{ s.t. } \overline{\dim}_B(\Gamma) \leqslant \beta\} \subseteq \{\text{percolation}\}. \quad (2.8)$$

Therefore

$$\begin{aligned} P_p\big(\{\nexists \text{ crossing } \Gamma \text{ s.t. } \overline{\dim}_B(\Gamma) \leqslant \beta\} \cap \{\text{percolation}\}\big) \\ \leqslant \; P_p\big(\Lambda^c \cap \{\text{percolation}\}\big) \; = \; 0. \quad (2.9) \end{aligned}$$

It will follow in the proof that for the specified range of values of p we have $P_p(\text{percolation}) > 0$, and hence from (2.9) we may deduce (2.7).

Given $\beta > 1$, using Theorem 2.5 we can ensure, by taking values of p sufficiently close to 1, that whenever percolation occurs, then with probability 1 there exist crossings of dimension at most β. Of course, percolating paths with dimension greater than β may also exist simultaneously. The restriction to $M \geqslant 3$ will prove necessary for Lemma 2.16.

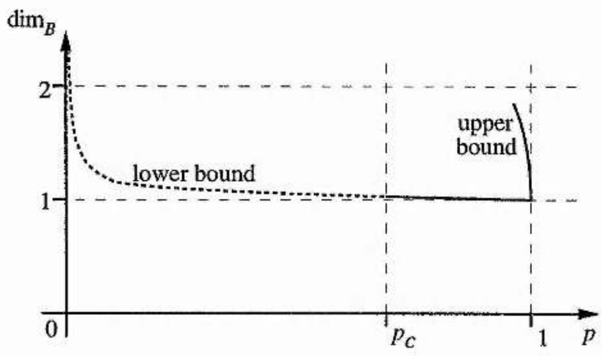


Figure 2.1: Upper and lower bounds on the box-counting dimension of crossings

The bounds presented above and illustrated by Figure 2.1 are by no means tight and could certainly be improved by the use of more careful estimates. Nevertheless, these are non-trivial bounds; for all $s > 4$, there exists a constant $v' = v'(s) > 0$ such that the lower bound is at least $1 + v'(1-p)^s$ for all $0 \leqslant p \leqslant 1$, whereas the upper bound behaves like $1 + 1/|\log(1-p)|$ as $p \to 1$.

We obtain the lower bound, Theorem 2.1, in Section 2.2 and the upper bound, Theorem 2.5, in Section 2.3. The lower bound also appears in Orzechowski [48].

2.2 Lower bound on lower box dimension

Our strategy will be to show that any percolating path Γ must avoid small 'holes' in C_∞ at infinitely many scales. To do this, we first define the notion of an m-long link. Roughly speaking, for $n \geqslant 1$, a link is the parallelogram between two nearby level-n squares; a link is said to be m-long if the pattern of squares therein at levels $(n+1)$ to $(n+m)$ contains sufficiently many holes so as to force any path passing through the link to make a large detour.

We show that these m-long links occur with high probability for sufficiently large values of m, and that in any chain of neighbouring level-m squares, such links appear with a certain degree of independence. Finally we prove that, for m sufficiently large and with probability 1, all chains contains at least a certain fixed proportion of m-long links, from which we obtain a lower bound on the dimension of any percolating paths.

We define a distance function on subsets of the unit square. For $A, B \subseteq [0,1]^2$, let

$$\operatorname{dist}(A, B) = \inf\{d_\infty(\mathbf{x}_1, \mathbf{x}_2) : \mathbf{x}_1 \in A, \mathbf{x}_2 \in B\} \tag{2.10}$$

where d_∞ is the metric on $\mathbb{R}^2$ given by taking the maximum difference in co-ordinates.

Fix $n \geqslant 1$ and let A and B be two level-n squares. Suppose that $\operatorname{dist}(A, B) = M^{-n}$; then without loss of generality we may assume (by rotating and reflecting as necessary) that B lies to the right of, and possibly below, A, as in one of three configurations shown in Figure 2.2. We then define the *link* $L(A, B)$ between

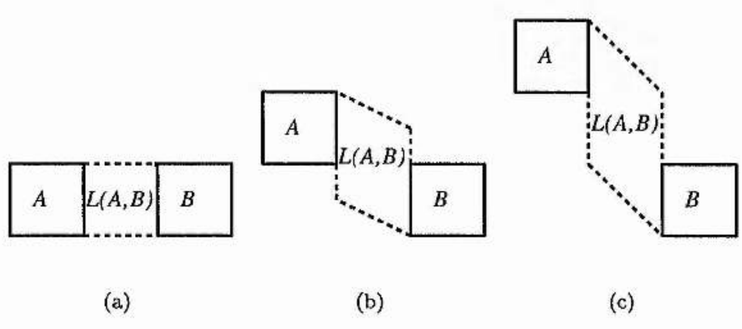


Figure 2.2: Defining the link $L(A, B)$

A and B to be the (closed) parallelogram formed by the intersection of the convex hull of A and B and the column of width M^{-n} separating A and B, as illustrated. (Note that if A and B are diagonally opposite one another, as in Figure 2.2c, then there are two possible choices for $L(A, B)$; choose either, in a consistent manner.)

Recall from page 25 the construction of the set $C_{n_1}^{n_2}$, defined only by the squares selected at levels n_1 through to n_2. Fix $n \geqslant 1$ and $m \geqslant 1$. We shall now define what it means for the link $L(A, B)$ between two level-n squares A, B to be m-long, in terms of the arrangement of squares selected at levels $(n+1)$ through to $(n+m)$. (Note therefore that this definition is independent of whether the squares A, B are actually contained in C_n or not.)

For points $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$, let $\ell(\mathbf{x}_1, \mathbf{x}_2)$ denote the straight-line segment joining $\mathbf{x}_1$ to $\mathbf{x}_2$. We say that $L(A, B)$ is *m-long* if for all $\mathbf{x}_1 \in A$ and $\mathbf{x}_2 \in B$, there exists a circle $c = c(\mathbf{x}_1, \mathbf{x}_2)$ such that:

$$
\begin{aligned}
&(i) && c \text{ is an open circle of diameter } M^{-(n+m)}/2 \\
&(ii) && c \subseteq L(A, B) \\
&(iii) && \text{The centre of } c \text{ lies on } \ell(\mathbf{x}_1, \mathbf{x}_2) \\
&(iv) && c \cap C_{n+1}^{n+m} = \varnothing.
\end{aligned}
\tag{2.11}
$$

Thus we think of $L(A, B)$ as m-long if for every pair of points $\mathbf{x}_1 \in A$, $\mathbf{x}_2 \in B$ there is a 'hole' in C_{n+1}^{n+m} of diameter at least $M^{-(n+m)}/2$ with centre on the line joining $\mathbf{x}_1$ to $\mathbf{x}_2$, as illustrated in Figure 2.3.

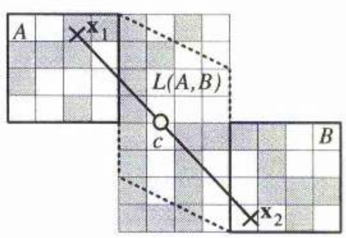


Figure 2.3: A 1-long link $L(A, B)$

LEMMA 2.6: Let A and B be level-n squares such that $\operatorname{dist}(A, B) = M^{-n}$ and let $\rho = 1 - (1-p)^2$. Then for all $m \geqslant 1$, we have

$$
P_p\big(L(A, B) \text{ is } m\text{-long}\big) \geqslant 1 - M^{4(m+1)} \rho^{M^m}. \tag{2.12}
$$

Proof: Assume that A, B and $L(A,B)$ appear as shown in Figure 2.2a, 2.2b or 2.2c and let $m \geqslant 1$. Let A' and B' be level-$(n+m+1)$ squares such that $A' \subseteq A$ and $B' \subseteq B$, and let $\mathbf{a}$, $\mathbf{b}$ denote the centres of A' and B' respectively; observe that there are $\left(M^{2(m+1)}\right)^2$ ways of choosing such a pair A', B'. Let $S_m(A',B')$ be the set of level-$(n+m)$ squares S_i such that S_i contains a point $\mathbf{z}_i \in \ell(\mathbf{a},\mathbf{b}) \cap L(A,B)$ of the form $\left((k+\frac{1}{2})M^{-(n+m)}, y\right)$, where $k \in \mathbb{Z}$ and $y \in [0,1]$.

For each $S_i \in S_m(A',B')$, let S_i' be the level-$(n+m)$ square vertically adjacent to S_i and closer to $\mathbf{z}_i$ (or either such square if both are equal distance from $\mathbf{z}_i$). Define the set of rectangles $R_m(A',B')$ by

$$R_m(A',B') = \left\{R_i : R_i = S_i \cup S_i',\ S_i \in S_m(A',B')\right\};$$

then $\mathrm{card} R_m(A',B') = M^m$ since there are $M^{-n}/M^{-(n+m)} = M^m$ distinct points $\mathbf{z}_i$ of the required form.

We declare a rectangle $R_i \in R_m(A',B')$ to be *vacant* if $\mathrm{int}(R_i) \cap C_{n+1}^{n+m} = \varnothing$, an event which occurs independently for each R_i with probability at least $(1-p)^2$. Observe that given a vacant rectangle $R_i \in R_m(A',B')$, then by our construction R_i contains an open circle $c(\mathbf{x}_1,\mathbf{x}_2)$ of diameter $M^{-(n+m)}/2$ and with centre lying on $\ell(\mathbf{x}_1,\mathbf{x}_2)$ such that $c \cap C_{n+1}^{n+m} = \varnothing$ for *all* points $\mathbf{x}_1 \in A'$ and $\mathbf{x}_2 \in B'$. We deduce that $L(A,B)$ is m-long if there exists a vacant $R_i \in R_m(A',B')$ for every pair of level-$(n+m+1)$ squares $A' \subseteq A$, $B' \subseteq B$. Therefore

$$P_p\left(L(A,B) \text{ is not } m\text{-long}\right) \leqslant \left(M^{2(m+1)}\right)^2 \rho^{M^m} \tag{2.13}$$

where $\rho = 1-(1-p)^2$.

■

LEMMA 2.7: Given $\varepsilon > 0$ and level-n squares A, B with $\mathrm{dist}(A,B) = M^{-n}$, let m_0 be the least integer m such that $P_p(L(A,B) \text{ is } m\text{-long}) \geqslant 1-\varepsilon$. Then m_0 satisfies

$$M^{m_0} \leqslant 2M\left(-\log\varepsilon + 4\log 8M - 8\log(1-p)\right)/(1-p)^2 \tag{2.14}$$

Proof: First suppose that m satisfies

$$M^m \geqslant 2(\log \varepsilon - 4k \log M)/\log \rho \tag{2.15}$$

where $k = 1 - \big(\log(1-\rho) - \log 8\big)/\log M$ and $\rho = 1-(1-p)^2$. Then

$$\begin{aligned} 4M^{1-k} = (1-\rho)/2 &\leqslant -(\log \rho)/2 && (2.16)\\ \Rightarrow \quad M^m(4M^{1-k} + \log \rho) &\leqslant M^m(\log \rho)/2 \\ &\leqslant \log \varepsilon - 4k \log M && (2.17) \end{aligned}$$

by (2.15). Hence

$$\begin{aligned} 4M^{m+1-k} + 4k \log M + M^m \log \rho &\leqslant \log \varepsilon && (2.18)\\ \Rightarrow \quad 4 \log M^{m+1} + M^m \log \rho &\leqslant \log \varepsilon && (2.19) \end{aligned}$$

since $\log x \leqslant x$ for $x > 0$. We take m to be the least integer value for which (2.15) holds; thus

$$\begin{aligned} M^m &< M.\,2(\log \varepsilon - 4k \log M)/\log \rho \\ &\leqslant 2M(-\log \varepsilon + 4k \log M)/(1-\rho) \\ &= 2M\big(-\log \varepsilon + 4(\log 8M - \log(1-\rho))\big)/(1-\rho) \\ &= 2M\big(-\log \varepsilon + 4 \log 8M - 8 \log(1-p)\big)/(1-p)^2. && (2.20) \end{aligned}$$

By (2.19) and Lemma 2.6, we have that $P_p\big(L(A,B) \text{ is } m\text{-long}\big) \geqslant 1-\varepsilon$, and so we deduce that $m_0 \leqslant m$ and hence $M^{m_0} \leqslant M^m$ as required.

■

Define a *level-n chain of size t* to be a set $S^{(n)} = \{S_1, \ldots, S_t\}$ of level-n squares satisfying

(i) $S_1 \cap L \neq \varnothing$

(ii) $S_t \cap R \neq \varnothing$

(iii) $S_i \cap S_{i+1}$ is either a singleton or an edge, for all $1 \leqslant i \leqslant t-1$

(iv) $S_i \cap S_j = \varnothing$ whenever $j \neq i \pm 1$. (2.21)

Fix $m \geqslant 1$. Given a chain $S^{(n)} = \{S_1, \ldots, S_{t+2}\}$ of size $t+2$ and $1 \leqslant i \leqslant t$, let $L(i)$ denote the event $\{L(S_i, S_{i+2}) \text{ is } m\text{-long}\}$. The events $L(i)$ are not in general independent, since if $L(S_i, S_{i+2})$ and $L(S_j, S_{j+2})$ have any squares in common at levels $(n+1)$ to $(n+m)$, knowledge of whether $L(i)$ occurs will have an effect on $L(j)$. However, with this definition of a chain, we can ensure that $L(S_i, S_{i+2})$ and $L(S_j, S_{j+2})$ do *not* touch, providing that $|i-j| \geqslant 4$.

LEMMA 2.8: The events $L(4), L(8), \ldots, L(4[t/4])$ (where $[x]$ denotes the integer part of x) are mutually independent.

Proof: Let i and j be integer multiples of 4 such that $4 \leqslant i < j \leqslant 4[t/4]$. Consider the four squares S_i, S_{i+2}, S_j and S_{j+2}. Since $S^{(n)} = \{S_1, \ldots, S_{t+2}\}$ is a chain, (2.21.*iii*) and (2.21.*iv*) imply that

$$\begin{aligned} \operatorname{dist}(S_i, S_{i+2}) &= M^{-n} & \operatorname{dist}(S_j, S_{j+2}) &= M^{-n} \\ \operatorname{dist}(S_i, S_j) &\geqslant M^{-n} & \operatorname{dist}(S_i, S_{j+2}) &\geqslant M^{-n} \\ \operatorname{dist}(S_{i+2}, S_j) &\geqslant M^{-n} & \operatorname{dist}(S_{i+2}, S_{j+2}) &\geqslant M^{-n}. \end{aligned} \tag{2.22}$$

The effect of the rules in (2.22) is to ensure that the squares S_i, S_{i+2}, S_j, S_{j+2} are mutually separated by a distance of at least M^{-n}; a few of the arrangements possible are illustrated in Figure 2.4. It is not hard to see that unless the four squares are arranged as in Figure 2.4d or Figure 2.4e (or a congruent arrangement) then we must have

$$\operatorname{dist}\big(L(S_i, S_{i+2}), L(S_j, S_{j+2})\big) \geqslant M^{-n}/2. \tag{2.23}$$

In this case, $L(S_i, S_{i+2})$ and $L(S_j, S_{j+2})$ have no squares in common at levels $(n+1)$ to $(n+m)$, and so we conclude that $L(i)$ and $L(j)$ are independent events.

Those configurations illustrated in Figure 2.4d and Figure 2.4e may be discounted since in each case the positioning of the squares S_{i+1} and S_{j+1} would contravene condition (2.21.*iv*) of the definition of a chain.

■

Define $\#S^{(n)} = \operatorname{card} S^{(n)} = t+2$ and

$$\#_L S^{(n)} = \operatorname{card}\big\{i : L(4i) \text{ occurs}, 1 \leqslant i \leqslant [t/4]\big\}.$$

LEMMA 2.9: Suppose that we have $P_p\big(L(A,B) \text{ is } m\text{-long}\big) \geqslant 1 - \varepsilon$ (where $0 < \varepsilon < 1$) for all pairs of level-n squares A, B with $\operatorname{dist}(A,B) = M^{-n}$, and for

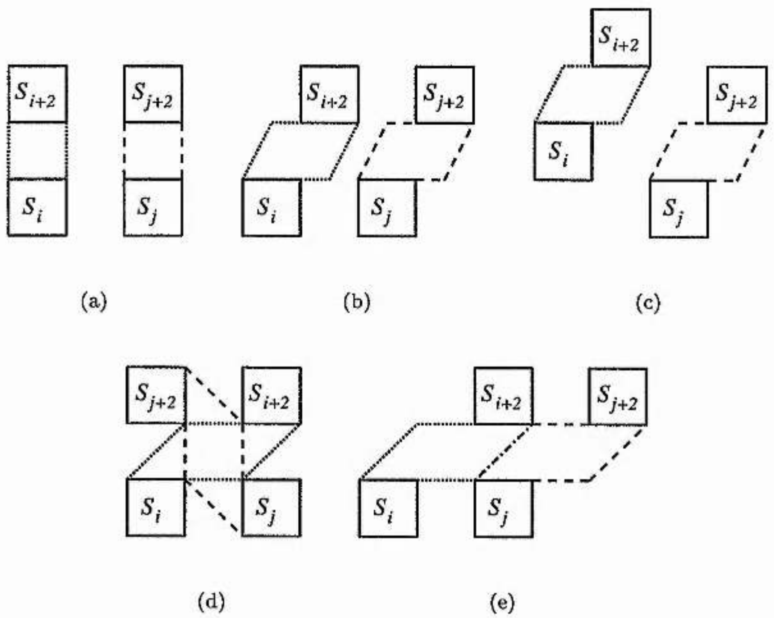


Figure 2.4: Possible arrangements of S_i, S_{i+2}, S_j, S_{j+2}

all $n \geqslant 1$. Let $\varepsilon \leqslant \xi < 1$. Then there exists a constant $c_1 = c_1(\varepsilon, \xi) > 0$ such that

$$P_p\big(\exists \text{ a level-}n \text{ chain } S^{(n)} \text{ s.t. } \#_L S^{(n)} \leqslant (1-\xi)[(\#S-2)/4]\big) \leqslant c_1 M^{2n}\big(5h(\varepsilon,\xi)^{1/4}\big)^{M^n} \tag{2.24}$$

for all $n \geqslant 1$, where $h(\varepsilon, \xi) = \left(\frac{\varepsilon}{\xi}\right)^{\xi}\left(\frac{1-\varepsilon}{1-\xi}\right)^{1-\xi}$.

Proof: Fix $n \geqslant 1$. Let $S^{(n)} = \{S_1, \ldots, S_{t+2}\}$ be a fixed level-n chain of size $t+2$, and define $N = [t/4]$. Let X be a random variable having the binomial distribution $\text{Bin}(N, 1-\varepsilon)$. Then

$$P_p(\#_L S^{(n)} \leqslant l) \leqslant P(X \leqslant l) \tag{2.25}$$

for all values of l, since by Lemma 2.8, $L(4), \ldots, L(4N)$ are mutually independent events, each occurring with probability at least $1-\varepsilon$. Now

$$P(X \leqslant l) = \sum_{j=0}^{l} \binom{N}{j} (1-\varepsilon)^j \varepsilon^{N-j} \tag{2.26}$$

and $\binom{N}{j}(1-\varepsilon)^j\varepsilon^{N-j}$ is increasing in j for $j \leqslant \mathbf{E}(X) = (1-\varepsilon)N$, so

$$P(X \leqslant l) \leqslant (l+1)\binom{N}{l}(1-\varepsilon)^l \varepsilon^{N-l} \tag{2.27}$$

provided that $l \leqslant (1-\varepsilon)N$. Using Stirling's formula $x! \approx \sqrt{2\pi}e^{-x}x^{x+1/2}$, it is easy to show that there exists a constant $c_0 > 0$, independent of N and l, such that

$$\binom{N}{l} \leqslant c_0 \left(\frac{N}{l(N-l)}\right)^{1/2} \left(\frac{N}{l}\right)^{l} \left(\frac{N}{N-l}\right)^{N-l} \tag{2.28}$$

for all N and l. Hence we deduce from (2.25), (2.27) and (2.28) with $l = [(1-\xi)N] < (1-\varepsilon)N$ that

$$\begin{aligned} P_p\big(\#_L S^{(n)} \leqslant (1-\xi)N\big) &\leqslant P\big(X \leqslant (1-\xi)N\big) = P(X \leqslant l) \\ &\leqslant (l+1)\binom{N}{l}(1-\varepsilon)^l \varepsilon^{N-l} \\ &\leqslant c_0 N \left(\frac{1}{N\xi(1-\xi)}\right)^{1/2} \left(\frac{1-\varepsilon}{1-\xi}\right)^{(1-\xi)N} \left(\frac{\varepsilon}{\xi}\right)^{\xi N} \\ &= c_0\big(N/\xi(1-\xi)\big)^{1/2}\big(h(\varepsilon,\xi)\big)^N \end{aligned} \tag{2.29}$$

where $h(\varepsilon,\xi) = \left(\frac{\varepsilon}{\xi}\right)^{\xi}\left(\frac{1-\varepsilon}{1-\xi}\right)^{1-\xi}$.

Next we count the number of level-n chains $S^{(n)}$. There are M^n possible choices for S_1, namely the level-n squares intersecting $\{0\} \times [0,1]$. Each square S_i gives rise to at most five choices for its successor S_{i+1} under the restriction that S_{i+1} and S_{i-1} do not touch, and hence there are at most $M^n 5^{t+1}$ chains of size $t+2$. Also observe that any level-n chain necessarily has size lying between M^n and M^{2n}. We deduce that

$$\begin{aligned} &P_p\big(\exists \text{ level-}n \text{ chain } S^{(n)} \text{ s.t. } \#_L S^{(n)} \leqslant (1-\xi)[(\#S-2)/4]\big) \\ \leqslant\ &\sum_{t=M^n-2}^{M^{2n}-2} P_p\big(\exists \text{ level-}n \text{ chain } S^{(n)} \text{ of size } t+2 \text{ s.t. } \#_L S^{(n)} \leqslant (1-\xi)[t/4]\big) \\ \leqslant\ &\sum_{t=M^n-2}^{M^{2n}-2} M^n 5^{t+1} c_0\big([t/4]/\xi(1-\xi)\big)^{1/2}\big(h(\varepsilon,\xi)\big)^{[t/4]} \end{aligned} \tag{2.30}$$

by (2.29). If $h(\varepsilon,\xi) < 5^{-4}$, then we can find a constant $c_1 = c_1(\varepsilon,\xi) > 0$ such that the last sum is at most $c_1 M^{2n}\big(5h(\varepsilon,\xi)^{1/4}\big)^{M^n}$ for all $n \geqslant 1$. If $h(\varepsilon,\xi) \geqslant 5^{-4}$, then (2.24) is satisfied simply by taking $c_1 = 1$.

■

Let $\Lambda_m(n,\lambda)$ denote the event $\{\exists \text{ level-}n \text{ chain } S^{(n)} \text{ s.t. } \#_L S^{(n)} \leqslant \lambda \#S^{(n)}\}$.

COROLLARY 2.10: Suppose that we are given ε and ζ satisfying $0 < \varepsilon < \zeta < 1$, $h(\varepsilon,\zeta) < 5^{-4}$ and $P_p\big(L(A,B) \text{ is } m\text{-long}\big) \geqslant 1-\varepsilon$ for all pairs of level-n squares

A, B with $\operatorname{dist}(A,B) = M^{-n}$, for all $n \geqslant 1$. Then

$$P_p\big(\Lambda_m(n,\lambda) \text{ occurs for infinitely many } n\big) = 0 \tag{2.31}$$

where $\lambda = (1-\zeta)/4$.

Proof: Since $h(\varepsilon,\zeta) < 5^{-4}$, we can find $\varepsilon < \xi < \zeta$ such that $h(\varepsilon,\xi) < 5^{-4}$. Applying Lemma 2.9 with this value of ξ, observing that

$$(1-\xi)[(t-2)/4] > (1-\zeta)t/4 \tag{2.32}$$

for all sufficiently large values of t, and summing over n yields

$$\sum_{n\geqslant 1} P_p\big(\Lambda_m(n,(1-\zeta)/4)\big) < \infty. \tag{2.33}$$

We then apply the Borel–Cantelli Lemma to deduce the desired result.

■

For $n \geqslant 1$, define a *level-n crossing* to be a continuous path $\gamma^{(n)}\colon [0,1] \to C_n$ such that $\gamma^{(n)}(0) \in L$ and $\gamma^{(n)}(1) \in R$. Observe that if percolation occurs in C_n then there exists a piecewise linear level-n crossing, which therefore has finite length. We shall let $\mathcal{L}(\gamma^{(n)})$ denote the length of a crossing $\gamma^{(n)}$.

Fix a set C_∞ and define $\Delta^{(n)} = \inf\{\mathcal{L}(\gamma^{(n)}) : \gamma^{(n)} \text{ is a level-}n\text{ crossing}\}$, with the convention that $\inf \varnothing = \infty$. Given a percolating path $\Gamma\colon [0,1] \to C_\infty$, recall that $N^{(n)}(\Gamma)$ denotes the smallest number of level-n squares that cover $\Gamma([0,1])$. By (2.1), the lower box dimension of a path $\Gamma \equiv \Gamma([0,1])$ is

$$\underline{\dim}_B(\Gamma) = \liminf_{n\to\infty} \frac{\log N^{(n)}(\Gamma)}{\log M^n}. \tag{2.34}$$

In addition, we define

$$N^{(n)} = \inf\big\{N^{(n)}(\Gamma) : \Gamma \text{ is a percolating path}\big\},$$

again with the convention that $\inf \varnothing = \infty$.

Observe now that $\Delta^{(n)} \leqslant \sqrt{2}M^{-n}N^{(n)}$ for all $n \geqslant 1$, since every level-n square can be crossed by a portion of path of length at most $\sqrt{2}M^{-n}$. Hence

$$\begin{aligned}\underline{\dim}_B(\Gamma) &\geqslant \liminf_{n\to\infty} \frac{\log(M^n\Delta^{(n)}/\sqrt{2})}{\log M^n} \\ &= 1 + \liminf_{n\to\infty} \frac{\log \Delta^{(n)}}{\log M^n}.\end{aligned} \tag{2.35}$$

The following geometrical result shows that if $\Lambda_m(n, \lambda)$ does not occur, *i.e.* for every level-n chain the proportion of m-long links is greater than λ, then the shortest level-$(n+m)$ crossing has length at least r times the length of the shortest level-n crossing, where $r = r(m) > 1$.

LEMMA 2.11: Let $m \geqslant 1$, $n \geqslant 1$ and $\lambda > 0$ and suppose that $\Lambda_m(n, \lambda)$ does not occur. Then

$$\Delta^{(n+m)} \geqslant \Delta^{(n)}\Big(1 + \frac{\lambda M^{-2m}}{288\sqrt{2}}\Big). \tag{2.36}$$

Proof: If there are no level-$(n+m)$ crossings then (2.36) is trivially satisfied. Otherwise, since C_{n+m} is a union of equal closed squares, there exists a crossing $\gamma^{(n+m)}$ such that $\mathcal{L}(\gamma^{(n+m)}) = \Delta^{(n+m)}$. Let

$$T^{(n)} = \{A\colon A \subseteq C_n \text{ is a level-}n \text{ square s.t. } A \cap \gamma^{(n+m)} \neq \varnothing\}.$$

By construction, $\bigcup_{A\in T^{(n)}} A$ is a connected subset of C_n intersecting both L and R. Hence by deleting squares as necessary so as to satisfy condition (2.21.*iv*), we can find $S^{(n)} = \{S_1, \dots, S_t\} \subseteq T^{(n)}$ such that $S^{(n)}$ is a level-n chain.

Since $\Lambda_m(n, \lambda)$ does not occur, we have that $\#_L S^{(n)} > \lambda \# S^{(n)}$ and so the set $I = \{i : L(4i) \text{ occurs}, 1 \leqslant i \leqslant [(t-2)/4]\}$ has cardinality greater than λt. Let $i \in I$; thus the link $L(S_{4i}, S_{4i+2})$ is m-long.

Fix points $\mathbf{y}_1 \in \gamma^{(n+m)} \cap S_{4i}$ and $\mathbf{y}_2 \in \gamma^{(n+m)} \cap S_{4i+2}$. For $l \leqslant n+m$, define $\gamma_{1,2}^{(l)}$ to be a path $\gamma_{1,2}^{(l)}\colon [0,1] \to C_l$ of minimal length such that $\gamma_{1,2}^{(l)}(0) = \mathbf{y}_1$ and $\gamma_{1,2}^{(l)}(1) = \mathbf{y}_2$ and let $\delta_i^{(l)} = \mathcal{L}(\gamma_{1,2}^{(l)})$. According to the relative position of $\mathbf{y}_1$ and $\mathbf{y}_2$ and the presence or absence of neighbouring level-n squares, we can pick $\gamma_{1,2}^{(n)}$ to appear like one of the eight possibilities shown in Figure 2.5.

Next define points $\mathbf{x}_k$ for $k = 1, 2$ by $\mathbf{x}_k = \gamma_{1,2}^{(n)}(s_k)$, where

$$s_1 = \sup\{s\colon \gamma_{1,2}^{(n)}(s) \in S_{4i}\} \quad \text{and} \quad s_2 = \inf\{s\colon \gamma_{1,2}^{(n)}(s) \in S_{4i+2}\};$$

then the portion of the path $\gamma_{1,2}^{(n)}$ between $\mathbf{x}_1$ and $\mathbf{x}_2$ is a straight-line segment. Since $L(S_{4i}, S_{4i+2})$ is m-long, there exists a circle $c = c(\mathbf{x}_1, \mathbf{x}_2)$ satisfying the conditions of (2.11) with $A = S_{4i}$, $B = S_{4i+2}$. Then because $C_{n+m} \subseteq C_{n+1}^{n+m}$ and $\gamma^{(n+m)}$ was chosen to be of minimal length, we see that $\gamma_{1,2}^{(n+m)}$ must 'go around the outside of the hole' c, as in Figure 2.6a.

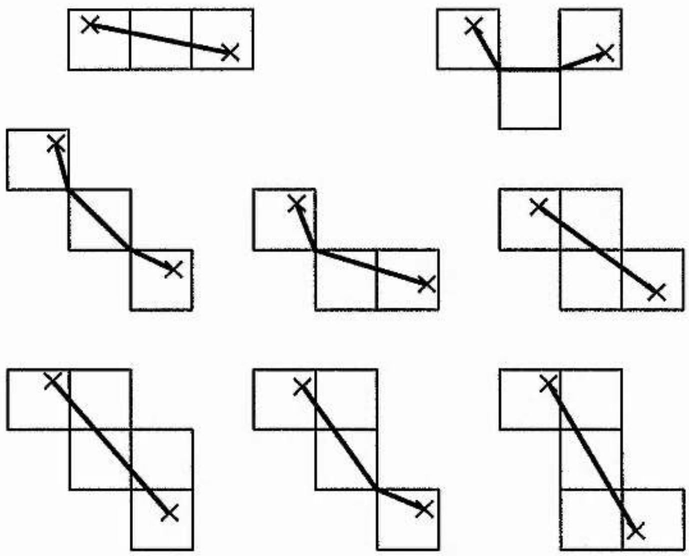


Figure 2.5: Possibilities for the path $\gamma_{0,1}^{(n)}$

Considering the 'worst case' shown in Figure 2.6b, where $\mathbf{y}_1$ and $\mathbf{y}_2$ are diagonally opposite and $\delta_i^{(n)} = 3\sqrt{2}M^{-n}$, we have

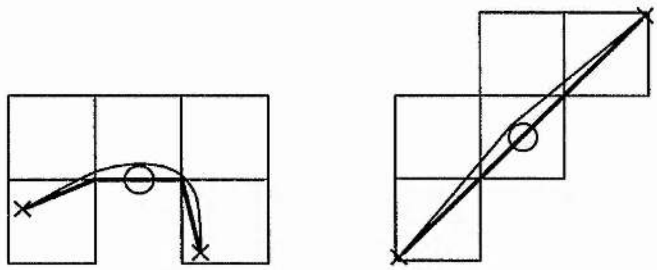
$$\begin{aligned} \delta_i^{(n+m)} &\geqslant \left((3\sqrt{2}M^{-n})^2 + (M^{-(n+m)}/2)^2\right)^{1/2} \\ &= 3\sqrt{2}M^{-n}(1 + M^{-2m}/72)^{1/2} \\ &\geqslant 3\sqrt{2}M^{-n}(1 + M^{-2m}/288) \end{aligned} \tag{2.37}$$

using $(1+x)^{1/2} \geqslant 1 + x/4$ for $0 \leqslant x \leqslant 1$. Since the ratio $\delta_i^{(n+m)}/\delta_i^{(n)}$ is certainly no less in other configurations, we conclude that for general $\mathbf{y}_1$ and $\mathbf{y}_2$ we have

$$\delta_i^{(n+m)} \geqslant \delta_i^{(n)}(1 + M^{-2m}/288). \tag{2.38}$$

For every $i \in I$, we now replace the portion of $\gamma^{(n+m)}$ between $\mathbf{y}_1 = \mathbf{y}_1(i)$ and $\mathbf{y}_2 = \mathbf{y}_2(i)$ by a path of the form $\gamma_{1,2}^{(n)}$, as above. For each i, this replacement causes a reduction in $\mathcal{L}(\gamma^{(n+m)})$ of at least

$$\delta_i^{(n+m)} - \delta_i^{(n)} \geqslant \delta_i^{(n)} M^{-2m}/288 \geqslant M^{-(n+2m)}/288.$$



(a) (b)

Figure 2.6: $\gamma_{0,1}^{(n+m)}$ avoids the circle c

The resulting path is a level-n crossing, and therefore

$$\begin{aligned}\Delta^{(n)} &\leqslant \Delta^{(n+m)} - (\text{card}I)M^{-(n+2m)}/288 \\ &\leqslant \Delta^{(n+m)} - \frac{\lambda M^{-2m}}{288\sqrt{2}}\Delta^{(n)} \end{aligned} \tag{2.39}$$

since $\text{card}I \geqslant \lambda t$ and $\Delta^{(n)} \leqslant \sqrt{2}M^{-n}t$.

■

Finally, we complete the proof of Theorem 2.1. We first find a value of m so that sufficiently many of the level-n links are m-long for all but finitely many n. This value of m is then used in conjunction with Lemma 2.11 to obtain an almost sure lower bound on the dimension of any crossings.

Proof of Theorem 2.1:

Choose $0 < \varepsilon < \xi < 1$ satisfying $h(\varepsilon, \xi) < 5^{-4}$, where $h(\varepsilon, \xi)$ is as defined in Lemma 2.9. Throughout the remainder of this proof, we let m be the least integer such that $P_p(L(A, B) \text{ is } m\text{-long}) \geqslant 1 - \varepsilon$ for all pairs of level-n squares A, B with $\text{dist}(A, B) = M^{-n}$ and for all $n \geqslant 1$; Lemma 2.7 gives an upper bound on m.

By Corollary 2.10, we have

$$P_p\big(\Lambda_m(n, (1-\xi)/4) \text{ occurs for infinitely many } n\big) = 0. \tag{2.40}$$

Let $n_0 = \sup\{n\colon \Lambda_m(n, (1-\xi)/4) \text{ occurs}\} + 1$; then n_0 is random, taking a finite value almost surely. When $n_0 < \infty$, we have $\Delta^{(n_0)} \geqslant 1$ trivially, and for all $k \geqslant 0$,

$$\Delta^{(n_0+(k+1)m)} \geqslant \Delta^{(n_0+km)}(1 + \eta M^{-2m}) \tag{2.41}$$

by Lemma 2.11, where $\eta = (1-\xi)/1152\sqrt{2}$. Since $\Delta^{(n)}$ is non-decreasing in n, there exists $d > 0$ such that

$$\Delta^{(n)} \geqslant d(1 + \eta M^{-2m})^{n/m} \tag{2.42}$$

for all $n \geqslant 1$; we thus have an almost sure lower bound on the limiting scaling behaviour of the lengths of the level-n crossings as $n \to \infty$. By (2.35), with probability 1 every percolating path $\Gamma\colon [0,1] \to C_\infty$ satisfies

$$\begin{aligned} \underline{\dim}_B(\Gamma) &\geqslant 1 + \frac{\log(1+\eta M^{-2m})^{n/m}}{\log M^n} \\ &\geqslant 1 + \frac{\eta M^{-2m}}{2\log M^m} \end{aligned} \tag{2.43}$$

since $\log(1+x) \geqslant x/2$ for $0 < x < 1$.

Recall that by Lemma 2.7 we have $M^m \leqslant 2Mw/(1-p)^2$, where $w = w(M, p, \varepsilon)$ $= \log((8M)^4 \varepsilon^{-1}(1-p)^{-8})$; substituting this inequality into (2.43) yields

$$\underline{\dim}_B(\Gamma) \geqslant 1 + \frac{\eta(1-p)^4}{8M^2w^2\big(\log 2M + \log w - 2\log(1-p)\big)}. \tag{2.44}$$

As $p \to 1$, the denominator behaves asymptotically like $\big(-\log(1-p)\big)^3$, and hence we can find a constant $v = v(M, \varepsilon, \xi) > 0$ such that

$$\underline{\dim}_B(\Gamma) \geqslant 1 + v(1-p)^4|\log(1-p)|^{-3} \tag{2.45}$$

uniformly in p, thereby establishing Theorem 2.1. Note that we can obtain an optimal estimate for v by maximising $v = v(M, \varepsilon, \xi)$ over the permitted range of ε and ξ.

■

2.3 Upper bound on upper box dimension

Our approach here will be to show that providing p is sufficiently close to 1, nearly all of the squares present in C_n are retained in C_{n+m} (for some fixed m), and hence the shortest crossings of C_{n+m} are not too much longer than those of C_n. The first part of the argument is motivated by the original proof of the existence of non-trivial percolation; see Chayes *et al.* [8]. The second part is geometrical; together, these results establish that for p close to 1, C_∞ contains a crossing with upper box dimension bounded above by β, with positive probability. The third part strengthens this further; by using a branching process argument, we show that for p sufficiently close to 1, conditional on percolation occurring, C_∞ in fact contains crossings of dimension at most β almost surely.

We thus obtain an (almost sure) upper bound on the minimal upper box dimension of crossings within any realisation of C_∞ in which percolation occurs. Note that this does *not* however give that the minimal dimension of crossings is an almost sure constant across all percolating realisations of C_∞.

Let A be a level-n square, where $n \geqslant 0$. For $i \geqslant 1$, define the *i-children* of A to be the set $\mathcal{C}_i(A)$ of all level-$(n+i)$ squares contained within A. For $m \geqslant 1$, define the *m-family* of A, $\mathcal{F}_m(A)$, by $\mathcal{F}_m(A) = \bigcup_{i=1}^m \mathcal{C}_i(A)$.

Let $p < 1$. Throughout this section, we fix $m \geqslant 1$ and let

$$\mu = \mu(m) = \text{card } \mathcal{F}_m(A) = \sum_{i=1}^{m} M^{2i} = M^2(M^{2m}-1)/(M^2-1).$$

We shall say that a level-n square A is *selected* if $A \subseteq C'_n$ (C'_n is defined on page 2.1). Define A to be *$(m;1)$-full* if B is selected for at least $\mu - 1$ of the squares $B \in \mathcal{F}_m(A)$; thus

$$P_p\big(A \text{ is } (m;1)\text{-full}\big) = p^\mu + \mu p^{\mu-1}(1-p). \tag{2.46}$$

Next we define the notions of *$(m;k)$-fullness* by induction on k as follows:
(*i*) If B is selected for all $B \in \mathcal{F}_m(A)$, then we declare A to be $(m;k+1)$-full if B is $(m;k)$-full for at least $M^{2m}-1$ of the $B \in \mathcal{C}_m(A)$.
(*ii*) If B is selected for exactly $\mu - 1$ of the $B \in \mathcal{F}_m(A)$, then we declare A to be $(m;k+1)$-full if B is $(m;k)$-full for at least $M^{2m}-1$ of the $B \in \mathcal{C}_m(A)$, including all those B remaining in C_{n+m}.
Otherwise A is not $(m;k+1)$-full. Note that these definitions mirror those of m-fullness in Section 1.2.

Let $F_k^m(A)$ denote the event $\{A \text{ is } (m;k)\text{-full}\}$ and let $\phi_k = P_p(F_k^m(A))$; then for all $k \geqslant 1$ we have

$$\phi_{k+1} \geqslant p^\mu\big(\phi_k^{M^{2m}} + M^{2m}\phi_k^{M^{2m}-1}(1-\phi_k)\big) + \mu p^{\mu-1}(1-p)\phi_k^{M^{2m}-1}. \tag{2.47}$$

Observe that defining $\phi_0 = 1$ gives $\phi_1 = p^\mu + \mu p^{\mu-1}(1-p)$ as required, so (2.47) holds for all $k \geqslant 0$.

Let $F_\infty^m(A) = \bigcap_{k\geqslant 1} F_k^m(A)$ and let $\phi_\infty = P_p(F_\infty^m(A))$. We say that A is $(m;\infty)$*-full* if $F_\infty^m(A)$ occurs.

LEMMA 2.12: Let A be a level-n square, for some $n \geqslant 0$. Then $F_1^m(A) \supseteq F_2^m(A) \supseteq \ldots \supseteq F_\infty^m(A)$.

Proof: Clearly $F_2^m(A) \subseteq F_1^m(A)$. Now we use induction: Suppose that for every $B \in \mathcal{C}_m(A)$ we have $F_k^m(B) \subseteq F_{k-1}^m(B)$. If A is $(m;k+1)$-full, then either condition (i) or (ii) above holds; in each case, all of the $(m;k)$-full m-children of A are also $(m;k-1)$-full, and hence we deduce that A is $(m;k)$-full.

∎

Define a dynamical system $\{\tilde\phi_k\}_{k\geqslant 0}$ by $\tilde\phi_0=1$ and, for $k\geqslant 0$, $\tilde\phi_{k+1} = f_p(\tilde\phi_k)$ where $f_p\colon [0,1]\to[0,1]$ is given by

$$\begin{aligned} f_p(x) &= p^\mu\left(\mu x^{\mu-1} - (\mu-1)x^\mu\right) + \mu p^{\mu-1}(1-p)x^{\mu-1} && (2.48)\\ &= \mu p^{\mu-1}x^{\mu-1} - (\mu-1)p^\mu x^\mu. && (2.49)\end{aligned}$$

We aim to show that $\{\tilde\phi_k\}$ is dominated by $\{\phi_k\}$, *i.e.* $\phi_k \geqslant \tilde\phi_k$ for all $k\geqslant 0$, and hence that $\phi_\infty = \lim_{k\to\infty}\phi_k$ is no less than the fixed point of f.

LEMMA 2.13: Let $k\geqslant 0$. Suppose that $\phi_k \geqslant \tilde\phi_k$; then $\phi_{k+1}\geqslant\tilde\phi_{k+1}$.

Proof: Define $g(x) = x^{M^{2m}} + M^{2m}x^{M^{2m}-1}(1-x)$; then by (2.47) we have

$$\phi_{k+1} \geqslant p^\mu g(\phi_k) + \mu p^{\mu-1}(1-p)\phi_k^{M^{2m}-1}. \qquad (2.50)$$

Differentiating $g(x)$, we obtain

$$g'(x) = M^{2m}(M^{2m}-1)x^{M^{2m}-2}(1-x) > 0; \qquad (2.51)$$

so $g(x)$ is increasing in x and hence

$$\phi_{k+1} \geqslant p^\mu g(\tilde\phi_k) + \mu p^{\mu-1}(1-p)\phi_k^{M^{2m}-1} \qquad (2.52)$$

since $\phi_k \geqslant \tilde\phi_k$.

Now consider the function $h(x,\alpha) = x^\alpha + \alpha x^{\alpha-1}(1-x)$, where $0\leqslant x\leqslant 1$ and $\alpha\geqslant 1$. Taking logarithms and differentiating with respect to α, we obtain

$$\begin{aligned}\frac{\partial}{\partial\alpha}\log h(x,\alpha) &= \log x + \frac{1-x}{x+\alpha(1-x)}\\ &\leqslant (x-1)+(1-x) = 0 && (2.53)\end{aligned}$$

since $x + \alpha(1-x) \geqslant 1$. We deduce that, holding x constant, $h(x,\alpha)$ is non-increasing in α, and hence we have

$$g(\tilde{\phi}_k) = h(\tilde{\phi}_k, M^{2m}) \geqslant h(\tilde{\phi}_k, \mu) = \tilde{\phi}_k^{\mu} + \mu\tilde{\phi}_k^{\mu-1}(1-\tilde{\phi}_k) \tag{2.54}$$

since $\mu \geqslant M^{2m}$. Finally observe that by (2.48), we can write $\tilde{\phi}_{k+1} = f_p(\tilde{\phi}_k)$ as

$$\tilde{\phi}_{k+1} = p^{\mu}\big(\tilde{\phi}_k^{\mu} + \mu\tilde{\phi}_k^{\mu-1}(1-\tilde{\phi}_k)\big) + \mu p^{\mu-1}(1-p)\tilde{\phi}_k^{\mu-1}; \tag{2.55}$$

combining (2.52), (2.54) and (2.55) and using $\phi_k^{M^{2m}-1} \geqslant \tilde{\phi}_k^{\mu-1}$ establishes that $\phi_{k+1} \geqslant \tilde{\phi}_{k+1}$.

■

On differentiating (2.49), we obtain

$$\begin{aligned} f_p'(x) &= \mu(\mu-1)p^{\mu-1}x^{\mu-2} - \mu(\mu-1)p^{\mu}x^{\mu-1} \\ &= \mu(\mu-1)p^{\mu-1}x^{\mu-2}(1-px) > 0 \end{aligned} \tag{2.56}$$

and hence we observe that f_p is an increasing function with $f_p(0) = 0$ and $f_p(1) < 1$. The following lemma establishes that f_p has a fixed point in the interval $[1-\mu^2(1-p)^2, 1)$.

LEMMA 2.14: Suppose that $1 - \mu^{-5/2}/3 \leqslant p < 1$. Then $f_p(x) \geqslant 1 - \mu^2(1-p)^2$ for all $x \in [1-\mu^2(1-p)^2, 1]$.

Proof: This proof is similar to the calculations in Section 2 of Chayes *et al.* [8], correcting some slight errors in their arithmetic. Dekking and Meester [17] also developed techniques for showing the existence of a phase transition; however, their Lemma 3.4 is not quite sufficient for our purposes as it does not give bounds for the fixed point of f_p as a function of p as $p \to 1$.

Write $x = 1-\delta$ and observe that for $\delta \leqslant \mu^{-1}$ we have

$$\begin{aligned} \mu x^{\mu-1} - (\mu-1)x^{\mu} &\geqslant \mu\big(1-(\mu-1)\delta\big) - (\mu-1)\big(1-\mu\delta+\mu(\mu-1)\delta^2/2\big) \\ &= 1 - \mu(\mu-1)^2\delta^2/2, \end{aligned} \tag{2.57}$$

$$x^{\mu-1} \geqslant 1-(\mu-1)\delta. \tag{2.58}$$

Next write $p = 1-\varepsilon$ and observe that for $\varepsilon \leqslant \mu^{-1}$ we have

$$p^{\mu} \geqslant 1 - \mu\varepsilon + \mu(\mu-1)\varepsilon^2/2 - \mu(\mu-1)(\mu-2)\varepsilon^3/6, \tag{2.59}$$

$$\mu p^{\mu-1}(1-p) \geqslant \mu\varepsilon - \mu(\mu-1)\varepsilon^2. \tag{2.60}$$

Substituting (2.57)–(2.60) into the definition of f_p gives

$$\begin{aligned} f_p(1-\delta) &\geqslant 1-\mu\varepsilon+\mu(\mu-1)\varepsilon^2/2-\mu(\mu-1)(\mu-2)\varepsilon^3/6-\mu(\mu-1)^2\delta^2 \\ &\quad +\mu\varepsilon-\mu(\mu-1)\varepsilon^2-\mu(\mu-1)\varepsilon\delta \\ &\geqslant 1-2\mu(\mu-1)\varepsilon^2/3-\mu(\mu-1)^2\delta^2/2-\mu(\mu-1)\varepsilon\delta \\ &> 1-\mu^2(2\varepsilon^2/3+\varepsilon\delta+\mu\delta^2/2). \end{aligned} \tag{2.61}$$

In particular, when $\delta=\mu^2\varepsilon^2$ we have

$$\begin{aligned} f_p(1-\delta) &> 1-\mu^2\varepsilon^2(2/3+\mu^2\varepsilon+\mu^5\varepsilon^2/2) \\ &\geqslant 1-8\mu^2\varepsilon^2/9 \end{aligned} \tag{2.62}$$

since $\mu^{5/2}\varepsilon\leqslant 1/3$ (by the hypothesis of the lemma) and $\mu^2\varepsilon=(\mu^{5/2}\varepsilon)\mu^{-1/2}\leqslant 1/6$ (using $\mu\geqslant 4$). Finally since f_p is increasing we have that $f_p(x)\geqslant 1-\mu^2\varepsilon^2$ for all $x\in[1-\mu^2\varepsilon^2,1]$.

■

COROLLARY 2.15: For $1-\mu^{-5/2}/3\leqslant p<1$, we have $1-\mu^2(1-p)^2\leqslant\phi_\infty<1$.

Proof: By Lemma 2.14, we have that $\tilde{\phi}_k\geqslant 1-\mu^2(1-p)^2$ for all $k\geqslant 0$. Using Lemma 2.13 repeatedly gives $\phi_k\geqslant\tilde{\phi}_k$ for all $k\geqslant 0$, and by Lemma 2.12 we have that $\{\phi_k\}_{k\geqslant 0}$ is a non-increasing sequence in k. We conclude that the limit $\phi_\infty=\lim_{k\to\infty}\phi_k$ exists and is at least $1-\mu^2(1-p)^2$.

■

Note that we can use Corollary 2.15 to obtain an upper bound on the critical probability p_c: If A and A' are two adjacent $(1;1)$-full level-n squares, then the level-$(n+1)$ subsquares of $A\cup A'$ form a connected unit. Based on this observation, it is easy to see that if the unit square $[0,1]^2$ is $(1;k)$-full for all $k\geqslant 1$ then C_∞ is connected and hence percolation occurs. We thus have $p_c\leqslant 1-\mu^{-5/2}/3=1-M^{-5}/3$. (When $M=3$ this gives $p_c\leqslant 0.998\,63$, which is the bound implied by the calculations of Chayes *et al.* [8] and Falconer [21].)

Let Ψ be a fixed rectangular subset of $[0,1]^2$ of the form $\Psi=[x_0,x_1]\times[y_0,y_1]$, where x_0,x_1,y_0,y_1 are all integer multiples of $M^{-n'}$ for some $n'\geqslant 0$; let $L'=\{x_0\}\times[y_0,y_1]$, $R'=\{x_1\}\times[y_0,y_1]$ and fix $n\geqslant n'$. Define a *level-n edge-chain* *across* Ψ to be a set $S^{(n)} = \{S_1, \ldots, S_t\}$ of level-n squares satisfying

$$\begin{array}{ll} (i) & S_i \subseteq C_n \cap \Psi \text{ for all } 1 \leqslant i \leqslant t \\ (ii) & S_1 \cap L' \neq \varnothing, \quad S_i \cap L' = \varnothing \text{ for all } i > 1 \\ (iii) & S_t \cap R' \neq \varnothing, \quad S_i \cap R' = \varnothing \text{ for all } i < t \\ (iv) & S_i \cap S_{i+1} \text{ is an edge, for all } 1 \leqslant i \leqslant t-1 \\ (v) & S_i \cap S_j \text{ is no more than a singleton whenever } j \neq i, i \pm 1. \end{array} \tag{2.63}$$

Define a *crossing within* $S^{(n)}$ to be a continuous path $\gamma^{(n)}\colon [0,1] \to S^{(n)}$ such that $\gamma^{(n)}(0) \in L'$, $\gamma^{(n)}(1) \in R'$ and also $\gamma^{(n)}$ does not pass through any level-n vertices (*i.e.* points of the form (jM^{-n}, kM^{-n}) where $j, k \in \mathbb{Z}$). Given a level-n edge-chain $S^{(n)}$, let $\delta(S^{(n)}) = \inf\{\mathcal{L}(\gamma^{(n)})\colon \gamma^{(n)} \text{ is a crossing within } S^{(n)}\}$, where $\mathcal{L}$ denotes length.

Our next geometrical lemma shows that a level-n edge-chain $S^{(n)}$ of $(m;\infty)$-full squares necessarily contains a level-$(n+m)$ edge-chain $S^{(n+m)}$ of $(m;\infty)$-full squares such that $\delta(S^{(n+m)})/\delta(S^{(n)}) \leqslant 1 + 6/M$ (remember that $m \geqslant 1$ is fixed throughout).

LEMMA 2.16: Suppose that $M \geqslant 3$. Given $\Psi \subseteq [0,1]^2$, let $S^{(n)} = \{S_1, \ldots, S_t\}$ be a level-n edge-chain across Ψ such that S_i is $(m;\infty)$-full for all $1 \leqslant i \leqslant t$. Then there exist r_i, $1 \leqslant i \leqslant t$ and a level-$(n+m)$ edge-chain

$$S^{(n+m)} = \{S_{1,1}, \ldots, S_{1,r_1}, \ldots, S_{t,1}, \ldots, S_{t,r_t}\}$$

across Ψ such that

$$\begin{array}{ll} (i) & S_{i,j} \subseteq S_i \text{ for all } 1 \leqslant j \leqslant r_i \\ (ii) & S_{i,j} \text{ is } (m;\infty)\text{-full for all } S_{i,j} \in S^{(n+m)} \\ (iii) & \delta(S^{(n+m)}) \leqslant \delta(S^{(n)})(1 + 6/M). \end{array} \tag{2.64}$$

Proof: First observe that

$$\delta(S^{(n)}) \geqslant 2M^{-n}t/3, \tag{2.65}$$

with the lowest possible ratio of the length $\delta(S^{(n)})$ to the size t occurring when $S^{(n)}$ appears as in Figure 2.7. Given $\varepsilon > 0$, let $\gamma^{(n)}$ be a crossing within $S^{(n)}$ such

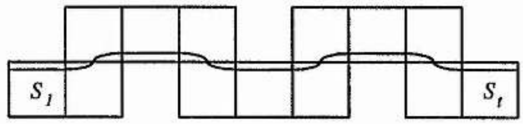


Figure 2.7: A crossing within $S^{(n)}$

that $\gamma^{(n)}([0,1]) \cap S_i$ is a connected set for all $1 \leqslant i \leqslant t$ and $\mathcal{L}(\gamma^{(n)}) \leqslant \delta(S^{(n)}) + \varepsilon$, noting that we can find such a crossing $\gamma^{(n)}$ for all $\varepsilon > 0$.

Take $1 \leqslant i \leqslant t$. Since S_i is $(m;\infty)$-full, there is at most one 'bad' subsquare of S_i, *i.e.* a square $B_i \in \mathcal{F}_m(S_i)$ that is not retained or a square $B_i \in \mathcal{C}_m(S_i)$ that is not $(m;\infty)$-full. Since $M \geqslant 3$, if such a square B_i exists then we can modify the portion of $\gamma^{(n)}$ contained in S_i so as to avoid touching B_i, yet still remain within S_i, by circumventing B_i as shown in Figure 2.8, at a cost of no more than $4M^{-(n+1)}$ in additional length.

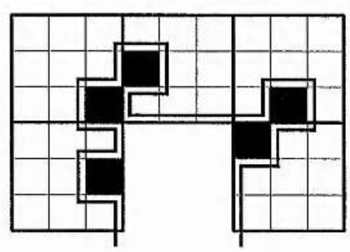


Figure 2.8: Modifying $\gamma^{(n)}$ to avoid the shaded squares B_i

Modifying $\gamma^{(n)}$ in this way for every B_i, $1 \leqslant i \leqslant t$, in turn (when such a square B_i exists) and ensuring that the path remains continuous gives a new crossing $\gamma^{(n+m)}$ such that

$$\begin{aligned} \mathcal{L}(\gamma^{(n+m)}) &\leqslant \mathcal{L}(\gamma^{(n)}) + 4M^{-(n+1)}t \\ &\leqslant \delta(S^{(n)})(1 + 6/M) + \varepsilon \end{aligned} \tag{2.66}$$

by (2.65). Finally we let $T^{(n+m)}$ be the set of level-$(n+m)$ squares intersecting $\gamma^{(n+m)}$ and then, by deleting squares and shortening $\gamma^{(n+m)}$ as necessary so as to satisfy (2.63.*ii*)–(2.63.*v*), we can find $S^{(n+m)} \subseteq T^{(n+m)}$ such that $S^{(n+m)}$ is a level-$(n+m)$ edge-chain across Ψ. This edge-chain satisfies (2.64.*i*) and (2.64.*ii*) by construction, and (2.64.*iii*) follows since there exist crossings $\gamma^{(n+m)}$ within

$S^{(n+m)}$ satisfying (2.66) for all $\varepsilon > 0$.

■

COROLLARY 2.17: Let $(A_1, \ldots, A_J)$, where $J \in \mathbb{N}$, be a horizontal row (respectively, vertical column) of adjacent level-n squares such that $A_i \subseteq C_n$ and A_i is $(m; \infty)$-full for all $1 \leqslant i \leqslant J$. Let $\Psi = \bigcup_{i=1}^{J} A_i$ and suppose that $M \geqslant 3$. Then there exists a path $\gamma: [0,1] \to C_\infty \cap \Psi$ crossing Ψ from left to right (respectively, top to bottom) such that $\overline{\dim}_B(\gamma) \leqslant 1 + \log 3/m \log M$.

Proof: Assume without loss of generality that $(A_1, \ldots, A_J)$ form a horizontal row. Let $S^{(n)} = \{A_1, \ldots, A_J\}$; then $S^{(n)}$ is a level-n edge-chain across Ψ. Applying Lemma 2.16 repeatedly, we deduce that for all $k \geqslant 0$ there exist edge-chains $S^{(n+km)}$ across Ψ such that

$$\delta(S^{(n+km)}) \leqslant \delta(S^{(n)})(1 + 6/M)^k. \tag{2.67}$$

Since $\delta(S^{(\tilde{n})})$ is non-decreasing in $\tilde{n}$, there exists $d > 0$ such that

$$\delta(S^{(\tilde{n})}) \leqslant d(1 + 6/M)^{\tilde{n}/m} \tag{2.68}$$

for all $\tilde{n} \geqslant n$.

For all $k \geqslant 0$, there exist crossings γ_k within $S^{(n+km)}$. Since the $\{S^{(n+km)}\}_{k\geqslant 0}$ are nested, we may inductively parametrise $\{\gamma_k\}_{k\geqslant 0}$ to ensure that

$$\left|\gamma_{k+1}(s) - \gamma_k(s)\right| \leqslant M^{-(n+km)} \tag{2.69}$$

for all $k \geqslant 0$ and $0 \leqslant s \leqslant 1$. To do this, given a parametrisation of γ_k, we divide up the interval in s corresponding to $\gamma_k(s) \in S_i$, where $S_i \in S^{(n+km)}$, amongst the level-$(n+(k+1)m)$ squares $S_{i,1}, \ldots, S_{i,r_i} \subseteq S_i$. We conclude that as $k \to \infty$, γ_k converges uniformly to a path $\gamma: [0,1] \to C_\infty \cap \Psi$ crossing Ψ from left to right.

Let $N^{(\tilde{n})}(\gamma)$ denote the number of level-$\tilde{n}$ squares intersecting γ, and observe that

$$N^{(\tilde{n})}(\gamma) \leqslant 3M^{\tilde{n}}\delta(S^{(\tilde{n})})/2 \tag{2.70}$$

for all $\tilde{n}$, as in (2.65). Hence by (2.2) and (2.68) we have

$$\overline{\dim}_B(\gamma) \quad = \quad \limsup_{\tilde{n}\to\infty} \frac{\log N^{(\tilde{n})}(\gamma)}{\log M^{\tilde{n}}}$$

$$\begin{aligned}
&\leqslant \limsup_{\tilde{n}\to\infty} \frac{\log 3M^{\tilde{n}}(1+6/M)^{\tilde{n}/m}}{2\log M^{\tilde{n}}} \\
&\leqslant 1+\frac{\log(1+6/M)}{m\log M} \\
&\leqslant 1+\frac{\log 3}{m\log M} \qquad (2.71)
\end{aligned}$$

since $M \geqslant 3$, as required.

■

Note that Lemma 2.16 and Corollary 2.17 are purely geometrical statements for a fixed value of m, without reference to the probability space.

We have so far shown that if $[0,1]^2$ is $(m;\infty)$-full then there exists a crossing with box dimension no greater than $1+\log 3/m\log M$; moreover if $p \geqslant 1-\mu^{-5/2}$ then $P_p\big([0,1]^2 \text{ is } (m;\infty)\text{-full}\big) \geqslant 1-\mu^2(1-p)^2$. We now go on to strengthen this statement to an almost sure result.

The next proposition establishes that, for $p \geqslant 1-\mu^{-5/2}/15$ and $n \geqslant 0$, the expected number of level-$(n+m)$ 'children' squares of a level-n square A that are not $(m;\infty)$-full is small, regardless of whether A itself is $(m;\infty)$-full or not.

PROPOSITION 2.18: Suppose that $p \geqslant 1-\mu^{-5/2}/15$ and let $n \geqslant 0$. Then for every level-n square A,

$$\mathbf{E}\big(\operatorname{card}\{B \in \mathcal{C}_m(A) \colon B \text{ is not } (m;\infty)\text{-full}\} \,\big|\, F_\infty^m(A)^c\big) \leqslant 1/10 \qquad (2.72)$$

and

$$\mathbf{E}\big(\operatorname{card}\{B \in \mathcal{C}_m(A) \colon B \text{ is not } (m;\infty)\text{-full}\} \,\big|\, F_\infty^m(A)\big) \leqslant 1/30. \qquad (2.73)$$

Proof: Fix $n \geqslant 0$ and a level-n square A and write $\mathcal{C}_m(A) = \{B_1, \ldots, B_{M^{2m}}\}$. Define the random variable $F = F(A)$ by

$$F(A) = \operatorname{card}\{i \colon F_\infty^m(B_i)^c \text{ occurs}\};$$

thus we wish to estimate the conditional expectation of $F(A)$, given the events $F_\infty^m(A)^c$ and $F_\infty^m(A)$ respectively. For a level-$\tilde{n}$ square B, $\tilde{n} \geqslant 0$, we let $Z(B)$ denote the event $\{B \subseteq C'_{\tilde{n}}\}$, *i.e.* $Z(B)$ occurs if B is selected. Observe that the events $\{Z(B_i)\}_{1\leqslant i\leqslant M^{2m}}$, $\{F_\infty^m(B_i)\}_{1\leqslant i\leqslant M^{2m}}$ are mutually independent with

$$P_p\big(Z(B_i)\big) = p \qquad (2.74)$$

$$P_p\big(F_\infty^m(B_i)\big) = \phi_\infty \qquad (2.75)$$

for all $1 \leqslant i \leqslant M^{2m}$. Since $p \geqslant 1 - \mu^{-5/2}/3$, we have $\phi_\infty \geqslant 1 - \mu^2(1-p)^2$ by Corollary 2.15.

In addition, for $1 \leqslant i \leqslant M^{2m}$ we define events

$$Q_i = \{Z(B_i)^c \cup F_\infty^m(B_i)^c\}$$

and the random variable $Q = Q(A)$ by

$$Q = \operatorname{card}\{i: Q_i \text{ occurs, } 1 \leqslant i \leqslant M^{2m}\}.$$

Then

$$\begin{aligned} P_p\big(F_\infty^m(B_i)^c \big| Q_i\big) &= P_p\big(F_\infty^m(B_i)^c\big)/P_p(Q_i) \\ &\leqslant \mu^2(1-p)^2/P_p(Q_i) \\ &= \mu^2(1-p)P_p\big(Z(B_i)^c\big)/P_p\big(Z(B_i)^c \cup F_\infty^m(B_i)^c\big) \\ &\leqslant \mu^2(1-p) \leqslant 1/30 \end{aligned} \tag{2.76}$$

since $\mu \geqslant 4$ and $p \geqslant 1 - \mu^{-5/2}/15$. Also since $F_\infty^m(B_i)^c \cap Q_i^c = \varnothing$ we have

$$P_p\big(F_\infty^m(B_i)^c \big| Q_i^c\big) = 0. \tag{2.77}$$

Now let $\mathcal{G}$ denote the σ-algebra generated by the events $\{Q_i: 1 \leqslant i \leqslant M^{2m}\}$ and let G be a set in $\mathcal{G}$. Since $F_\infty^m(B_i)$ and Q_j are independent for all $j \neq i$, if $G \subseteq Q_i$ then we have

$$P_p\big(F_\infty^m(B_i)^c \big| G\big) \leqslant 1/30 \tag{2.78}$$

by (2.76), and if $G \subseteq Q_i^c$ then we have

$$P_p\big(F_\infty^m(B_i)^c \big| G\big) = 0 \tag{2.79}$$

by (2.77). Summing over i, we obtain

$$\begin{aligned} \mathbf{E}(F|G) &= \sum_{i=1}^{M^{2m}} P_p\big(F_\infty^m(B_i)^c \big| G\big) \\ &\leqslant \operatorname{card}\{i: G \subseteq Q_i,\, 1 \leqslant i \leqslant M^{2m}\}/30 \\ &= Q/30 \end{aligned} \tag{2.80}$$

noting that Q is $\mathcal{G}$-measurable. Since (2.80) holds for all $G \in \mathcal{G}$, we have $\mathbf{E}(F|\mathcal{G}) \leqslant Q/30$.

Next, let $\mathcal{H}$ be the σ-algebra generated by $\{Z(B): B \in \mathcal{F}_m(A) \backslash \mathcal{C}_m(A)\}$. Since the fullness of A is determined by $Z(B)$ for $B \in \mathcal{F}_m(A)$ and $F_\infty^m(B)$ for

$B \in \mathcal{C}_m(A)$, we see that the event $F_\infty^m(A)$ is $\sigma(\mathcal{G},\mathcal{H})$-measurable. Also observe that $\mathcal{H}$ is independent of $\sigma\big(\sigma(F),\mathcal{G}\big)$, and therefore

$$\mathbf{E}\big(F \,\big|\, \sigma(\mathcal{G},\mathcal{H})\big) \;=\; \mathbf{E}(F|\mathcal{G}) \leqslant Q/30. \tag{2.81}$$

The iterated conditional expectations property, or 'tower' property, states that

$$\mathbf{E}(X|\mathcal{H}') = \mathbf{E}\big(\mathbf{E}(X|\mathcal{G}')\big|\mathcal{H}'\big) \tag{2.82}$$

when $\mathcal{H}'$ is a sub-σ-algebra of $\mathcal{G}'$ (see Williams [57], 9.7(i)). Applying (2.82) with $X = F$, $\mathcal{G}' = \sigma(\mathcal{G},\mathcal{H})$ and $\mathcal{H}' = \sigma(F_\infty^m(A))$, we have

$$\begin{aligned}\mathbf{E}\big(F\big|\sigma(F_\infty^m(A))\big) \;&=\; \mathbf{E}\big(\mathbf{E}(F|\sigma(\mathcal{G},\mathcal{H}))\,\big|\,\sigma(F_\infty^m(A))\big)\\ &\leqslant\; \mathbf{E}\big(Q/30\,\big|\,\sigma(F_\infty^m(A))\big)\end{aligned} \tag{2.83}$$

by (2.81).

Observe that for $j > 1$ we have the inclusions

$$\{Q = j\} \subseteq \{Q > 1\} \subseteq \{F_\infty^m(A)^c\}, \tag{2.84}$$

and so

$$P_p\big(Q = j \,\big|\, F_\infty^m(A)^c\big) \;=\; P_p\big(Q = j \,\big|\, Q > 1\big)P_p\big(Q > 1 \,\big|\, F_\infty^m(A)^c\big) \tag{2.85}$$

for all $j > 1$, and

$$P_p\big(Q = 1 \,\big|\, F_\infty^m(A)^c\big) \leqslant 1 - P_p\big(Q > 1 \,\big|\, F_\infty^m(A)^c\big). \tag{2.86}$$

Hence

$$\begin{aligned}\mathbf{E}\big(Q \,\big|\, F_\infty^m(A)^c\big) \;&=\; \sum_{j\geqslant 0} jP_p\big(Q = j \,\big|\, F_\infty^m(A)^c\big)\\ &\leqslant\; 1 - P_p\big(Q > 1 \,\big|\, F_\infty^m(A)^c\big) +\\ &\qquad\qquad \sum_{j>1} jP_p\big(Q = j \,\big|\, Q > 1\big)P_p\big(Q > 1 \,\big|\, F_\infty^m(A)^c\big)\\ &=\; 1 - P_p\big(Q > 1 \,\big|\, F_\infty^m(A)^c\big) +\\ &\qquad\qquad \mathbf{E}\big(Q \,\big|\, Q > 1\big)P_p\big(Q > 1 \,\big|\, F_\infty^m(A)^c\big)\\ &\leqslant\; \mathbf{E}(Q \,|\, Q > 1)\Big(1 - P_p\big(Q > 1 \,\big|\, F_\infty^m(A)^c\big) + P_p\big(Q > 1 \,\big|\, F_\infty^m(A)^c\big)\Big)\\ &=\; \mathbf{E}(Q \,|\, Q > 1)\end{aligned} \tag{2.87}$$

since $\mathbf{E}(Q|Q>1) \geqslant 1$. Now because $P_p(Q_i) = 1 - \phi_\infty p$, and the $\{Q_i\}$ are independent, Q has binomial distribution $\mathrm{Bin}(M^{2m}, 1-\phi_\infty p)$. Since $1 - \phi_\infty p$ is small, it is easy to see that

$$\mathbf{E}(Q \mid Q > 1) \leqslant 3; \tag{2.88}$$

taking $p \geqslant 1 - \mu^{-5/2}/15$ and $\phi_\infty \geqslant 1 - \mu^2(1-p)^2$ will certainly ensure that $1 - \phi_\infty p$ is sufficiently small. Combining (2.83), (2.87) and (2.88), we deduce that

$$\mathbf{E}\big(F \big| F_\infty^m(A)^c\big) \leqslant \mathbf{E}\big(Q \big| F_\infty^m(A)^c\big)/30 \leqslant \mathbf{E}(Q|Q>1)/30 \leqslant 1/10. \tag{2.89}$$

Finally note that, again from (2.83),

$$\mathbf{E}\big(F \big| F_\infty^m(A)\big) \leqslant \mathbf{E}\big(Q \big| F_\infty^m(A)\big)/30 \leqslant 1/30 \tag{2.90}$$

since $F_\infty^m(A) \subseteq \{Q \leqslant 1\}$.

■

Let $n \geqslant 0$ and let $A \equiv A^0$ be a level-n square; let $A^1, \dots, A^8$ denote the eight level-n squares neighbouring A. (If A touches the boundary of $[0,1]^2$, we take notional squares A^j lying outside the unit square as required, and declare any such notional square to be $(m;\infty)$-full). We define A to be *sealed* if A^j is $(m;\infty)$-full for all $0 \leqslant j \leqslant 8$. Observe then that if A is sealed, we may apply Corollary 2.17 to deduce that there exist paths of dimension at most

$$\beta = \beta(M, m) = 1 + \log 3/m \log M$$

in C_∞ crossing the rows and columns formed by those squares A^j which are present in C_n. We shall refer to such paths as the *sealing paths* of A, as illustrated in Figure 2.9. Later, these sealing paths will be used to modify any percolating path in C_∞ to produce crossings of dimension at most β.

For $0 \leqslant j \leqslant 8$, recall that $\mathcal{C}_m(A^j)$ represents the set of m-children of A^j. We define $\mathcal{C}'_m(A) = \bigcup_{j=0}^{8} \mathcal{C}_m(A^j)$ and let

$$F'(A) = \mathrm{card}\big\{B \in \mathcal{C}'_m(A) \colon B \text{ is not } (m;\infty)\text{-full}\big\}.$$

Define a process $\{Y_r\}_{r\in\mathbb{N}}$ and a sequence of sets of sets$\{H_r\}_{r\in\mathbb{N}}$ inductively as follows: If $[0,1]^2$ is not $(m;\infty)$-full, let $H_0 = \{[0,1]^2\}$ and $Y_0 = 1$; otherwise

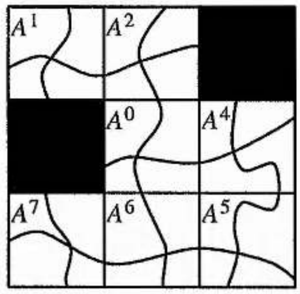


Figure 2.9: Sealing paths

let $H_0 = \varnothing$ and $Y_0 = 0$. For $r \geqslant 1$, let

$$H_r = \{B \in \mathcal{C}'_m(A) \colon A \in H_{r-1} \text{ and } B \text{ is not } (m;\infty)\text{-full}\}$$

and $Y_r = \operatorname{card} H_r$; thus Y_r is the total number of non-full m-children of the squares in H_r and their immediate neighbours. We think of $\{Y_r\}$ as being like a branching process, except that there is some dependence between the number of offspring of neighbouring squares.

Our next probabilistic lemma shows that for sufficiently high values of p, the process $\{Y_r\}_{r\geqslant 0}$ will eventually become extinct, almost surely.

LEMMA 2.19: Suppose that $p \geqslant 1 - \mu^{-5/2}/15$. With probability 1, there exists $0 \leqslant r_0 < \infty$ such that $Y_{r_0} = 0$.

Proof: For $r \geqslant 0$, let $\mathcal{F}_r$ be the σ-field generated by the events

$$\{F^m_\infty(A) \colon \text{level-}(sm) \text{ squares } A,\ 0 \leqslant s \leqslant r\}$$

and observe that each Y_r is then $\mathcal{F}_r$-measurable. By Proposition 2.18, for every level-(rm) square $A \equiv A^0$ we have

$$\begin{aligned}\mathbf{E}\big(F'(A)\big|\mathcal{F}_r\big) &= \sum_{j=0}^{8} \mathbf{E}\big(\operatorname{card}\{B \in \mathcal{C}^m(A^j) \colon B \text{ is not } (m;\infty)\text{-full}\}\,\big|\,\mathcal{F}_r\big) \\ &\leqslant 9/10. \end{aligned} \tag{2.91}$$

For $r \geqslant 0$ and a level-(rm) square A, $F'(A)$ is the number of non-full m-children of A and its immediate neighbours; thus we have

$$Y_{r+1} = \sum_{A \in H_r} F'(A). \tag{2.92}$$

Therefore

$$\mathbf{E}(Y_{r+1}|\mathcal{F}_r) = \sum_{A\in H_r} \mathbf{E}\big(F'(A)\big|\mathcal{F}_r\big) \leqslant 9Y_r/10. \tag{2.93}$$

Defining $Z_r = (10/9)^r Y_r$, we have

$$\mathbf{E}(Z_{r+1}|\mathcal{F}_r) \leqslant (10/9)^{r+1} 9Y_r/10 = Z_r \tag{2.94}$$

and $\mathbf{E}(|Z_r|) < \infty$, *i.e.* $\{Z_r\}_{r\geqslant 0}$ is a supermartingale. Since $\{Z_r\}$ is non-negative, by the supermartingale convergence theorem there exists a random variable Z such that $Z_r \to Z$ as $r \to \infty$ almost surely, and hence $Y_r \to 0$ almost surely. Finally, since $\{Y_r\}$ is integer-valued, we conclude that with probability 1, there exists $0 \leqslant r_0 < \infty$ such that $Y_{r_0} = 0$.

■

Note that the event $\{\exists r_0 \text{ s. t. } Y_{r_0} = 0\}$ is the event Λ of probability 1 referred to in (2.8). The next deterministic lemma establishes that when Λ occurs, every point of $[0,1]^2$ is contained in a sealed square.

LEMMA 2.20: Suppose that $Y_{r_0} = 0$ for some $0 \leqslant r_0 < \infty$. Then for all $\mathbf{x} \in [0,1]^2$, there exists $r = r(\mathbf{x})$ with $0 \leqslant r \leqslant r_0$ and a level-(rm) square $A(\mathbf{x})$ such that $\mathbf{x} \in A(\mathbf{x})$ and $A(\mathbf{x})$ is sealed.

Proof: Let $\Sigma_{-1} = [0,1]^2$ and, for $r \geqslant 0$, define the pointwise union

$$\Sigma_r = \bigcup_{A\in H_r} \bigcup_{j=0}^{8} A^j$$

where $A^0 \equiv A$ and $A^1, \ldots, A^8$ are the eight level-(rm) squares neighbouring A. Observe that every level-(rm) square $A \subseteq \overline{\Sigma_{r-1}\backslash\Sigma_r}$ is sealed, as otherwise one of the A^j, $0 \leqslant j \leqslant 8$, would belong to H_r.

Since $H_{r_0} = \varnothing$, we have $\Sigma_{r_0} = \varnothing$ and hence we may write

$$[0,1]^2 = \bigcup_{r=0}^{r_0} \Sigma_{r-1}\backslash\Sigma_r. \tag{2.95}$$

Thus every point $\mathbf{x} \in [0,1]^2$ is contained in $\Sigma_{r-1}\backslash\Sigma_r$ for some $0 \leqslant r \leqslant r_0$, and so there exists a level-(rm) square $A(\mathbf{x}) \ni \mathbf{x}$ such that $A(\mathbf{x}) \subseteq \overline{\Sigma_{r-1}\backslash\Sigma_r}$ and hence is sealed.

■

We now bring together our geometrical and our probabilistic results.

LEMMA 2.21: Suppose that $M \geqslant 3$ and $p \geqslant 1 - \mu^{-5/2}/15$. Then

$$P_p\big(\exists \text{ crossing } \Gamma \text{ s. t. } \overline{\dim}_B(\Gamma) \leqslant \beta \,\big|\, \text{percolation}\big) = 1$$

where $\beta = \beta(M, m) = 1 + \log 3/m \log M$.

Proof: First note that $1 - \mu^{-5/2}/15 > p_c$, and hence percolation occurs with non-zero probability; conditional on percolation occurring, by (1.25) there exists a percolating path with probability 1. Since there are only countably many points of the form (jM^{-n}, kM^{-n}), where $j, k \in \mathbb{Z}$ and $n \geqslant 0$, each of which will eventually be removed, almost surely, we may restrict our attention to percolating paths passing through no level-n vertices, for all n. Suppose that $\gamma: [0, 1] \to C_\infty$ is such a path. We shall modify γ piece by piece to obtain a new percolating path Γ with upper box dimension at most β.

Defining the process $\{Y_r\}_{r \geqslant 0}$ as above, by Lemma 2.19 there almost surely exists a finite $r_0 \geqslant 0$ such that $Y_{r_0} = 0$. Let

$$s = \sup\{t: \overline{\dim}_B(\gamma([0, t])) \leqslant \beta\}$$

and let $A = A_r(\gamma(s))$ be the sealed level-(rm) square identified by Lemma 2.20, taking $r = r(\gamma(s))$ minimal so that $0 \leqslant r \leqslant r_0$. Since A is sealed, we may apply Corollary 2.17 to the rows and columns formed by those squares A^j which are present in C_{rm}. We deduce that there are sealing paths of dimension at most β, lying in C_∞, crossing these rows and columns both from left to right and from top to bottom, as shown in Figure 2.9.

We can find points $\gamma(s_0)$ and $\gamma(s_1)$, where $s_0 < s < s_1$, lying on the intersection of the sealing paths with γ such that $\gamma([0, s_0] \cup [s_1, 1]) \cap A = \varnothing$ (or, if $\gamma(0) \in A$ take $s_0 = 0$, and if $\gamma(1) \in A$ take $s_1 = 1$). Remove the portion of path $\gamma([s_0, s_1])$ and substitute instead sections of the sealing paths, so as to ensure that the new γ remains a continuous percolating path. Note that we now have $\overline{\dim}_B(\gamma([0, s_1])) \leqslant \beta$.

We repeat this process until we have all of the original path has been replaced by sections of sealing path, observing that it must terminate in a finite number of steps when r_0 is finite. We conclude that

$$P_p\big(\exists \text{ crossing } \Gamma \text{ s.t. } \overline{\dim}_B(\Gamma) \leqslant \beta \,\big|\, \exists \text{ crossing } \gamma\big) = 1. \tag{2.96}$$

■

Finally, it remains to find the dependence of m on p.

Proof of Theorem 2.5:

First observe that if $m = 1$ then $\mu = \mu(1) = M^2$, and so the hypothesis $p \geqslant 1 - M^{-5}/15$ is equivalent to $p \geqslant 1 - \mu^{-5/2}/15$. By Lemma 2.21, we have

$$P_p\big(\exists \text{ crossing } \Gamma \text{ s. t. } \overline{\dim}_B(\gamma) \leqslant \beta \,\big|\, \text{percolation}\big) = 1$$

where $\beta = 1 + \log 3/\log M$.

We may improve this bound for larger values of p by taking $m > 1$; we wish to find the greatest integer m such that $p \geqslant 1 - \mu^{-5/2}/15$, where $\mu = \mu(m) = M^2(M^{2m} - 1)/(M^2 - 1)$. Solving for m gives

$$\begin{aligned} M^{2m} &= 1 + (1 - M^{-2})\mu & (2.97)\\ \Rightarrow \quad m &= \log\big(1 + (1 - M^{-2})\mu\big)/2\log M & \\ &\leqslant \frac{\log\big(1 + (1 - M^{-2})(15(1-p))^{-2/5}\big)}{2\log M}. & (2.98) \end{aligned}$$

We thus take

$$m = \max\Big(1, \Big[\log\big(1 + (1 - M^{-2})(15(1-p))^{-2/5}\big)/2\log M\Big]\Big) \tag{2.99}$$

(where $[x]$ denotes the integer part of x) in Lemma 2.21, observing that

$$\begin{aligned} m\log M &\geqslant \log\big(1 + (1 - M^{-2})(15(1-p))^{-2/5}\big)/2 - \log M & \\ &\geqslant \log\big((1 - M^{-2})^{1/2}(15(1-p))^{-1/5}M^{-1}\big). & (2.100) \end{aligned}$$

We conclude that, conditional on percolation occurring, with probability 1 there exists a crossing Γ such that $\overline{\dim}_B(\Gamma) \leqslant \beta$, where

$$\begin{aligned} \beta &= \beta(M, m) = 1 + \frac{\log 3}{m\log M} & \\ &\leqslant 1 + \frac{\log 3}{\log\big((1 - M^{-2})^{1/2}(15(1-p))^{-1/5}M^{-1}\big)} \quad \text{by (2.100)} & \\ &= 1 + \frac{\log 3}{\big|\big(\log(1-p)\big)/5 - u\big|} & (2.101) \end{aligned}$$

and $u = u(M) = \log\big((1 - M^{-2})^{-1/2}15^{1/5}M\big) > 0$.

■

Chapter 3

Percolation in Three and Higher Dimensions

In this chapter we shall consider the natural generalisation of the fractal percolation process in the plane to higher dimensional Euclidean space. The familiar concept of percolation translates to the d-dimensional cube as expected; we say that percolation occurs in the retained set $C_\infty \subset [0,1]^d$ if C_∞ contains a connected component intersecting opposite faces of $[0,1]^d$. However, when $d \geqslant 3$, the possibility of stronger forms of percolation arises; for example, C_∞ may contain subsets that are homeomorphic images of k-dimensional manifolds for some $2 \leqslant k \leqslant d-1$. We prove that at least some of these stronger forms can occur with non-zero probability, and that the limit set C_∞ passes through at least two distinct phase transitions as p is varied.

We shall pay particular attention to two notions of this stronger form of percolation; *sheet percolation* and, in the case $d = 3$, *disc percolation*. Full definitions are given later; briefly, sheet percolation occurs if the retained set contains a spanning set separating opposite faces of the cube, whilst disc percolation occurs if this spanning set is homeomorphic to a disc. Clearly, therefore, in three dimensions disc percolation implies sheet percolation; the converse is not the case, since we may add 'handles' to any disc, and remove holes where the handles have been joined, to produce a surface that is no longer a disc.

As for ordinary percolation, we are able to define critical probabilities p_s and p_d for the phase transitions to sheet percolation and disc percolation respectively. It is easily seen that $p_c \leqslant p_s \leqslant 1$ and, when $d = 3$, $p_s \leqslant p_d \leqslant 1$; in Section 3.2 we shall establish that $p_c < p_d < 1$. It is not at present known whether $p_s = p_d$ or $p_s < p_d$.

In Section 3.3 the transition to sheet percolation is examined in greater detail; we obtain a limiting value for p_s as the subdivision index M tends to infinity and use this to show that $p_c < p_s$ for sufficiently large values of M. Much of the material in Section 3.3 also appears in Orzechowski [47].

3.1 The d-dimensional random Cantor set

Let $d \geqslant 2$, $M \geqslant 2$ and $0 \leqslant p \leqslant 1$. We construct the *$d$-dimensional random Cantor set* C_∞ as follows: Write C_0 for the unit cube $[0,1]^d$ of $\mathbb{R}^d$. Divide C_0 into M^d equal closed subcubes, each of side-length M^{-1}, in the natural way; these are known as the *level-1* cubes. We retain each of these level-1 cubes independently at random with probability p, discarding those not retained, and write C_1 for the union of those remaining. Similarly, we next divide each level-1 cube of C_1 into M^d subcubes each of side-length M^{-2} and retain each of these *level-2* cubes independently at random with probability p, writing C_2 for the union of those retained. Continuing this process, we obtain a decreasing sequence of closed sets $C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$, with a well-defined limit $C_\infty = \bigcap_{n=0}^{\infty} C_n$. The natural induced probability measure is denoted by P_p. Where it is necessary to distinguish between the sets formed by differing values of the subdivision index M, we shall write the limit set as $C_\infty^{[M]}$.

We shall sometimes use a shorthand notation to refer to the faces or edges of the cube $[0,1]^3$ or square $[0,1]^2$: Let $I = [0,1]$, $\Delta = I^2 = [0,1]^2$, the left face $L = \{0\} \times I$ or $\{0\} \times \Delta$ (as applicable), the right face $R = \{1\} \times I$ or $\{1\} \times \Delta$, the bottom face $B = I \times \{0\}$ or $\Delta \times \{0\}$ and the top face $T = I \times \{1\}$ or $\Delta \times \{1\}$.

Analogously to the definition in two dimensions, we shall define *percolation* to occur in C_∞ if C_∞ contains a connected component intersecting both L and R. We let $\theta(p) = P_p(\text{percolation in } C_\infty)$ and define the critical probability $p_c = p_c(M,d) = \inf\{p : \theta(p) > 0\}$. By comparing the retained set in a single face of the cube $[0,1]^d$, for instance $C_\infty \cap B$, to the $(d-1)$-dimensional random Cantor set C'_∞ it is easy to see that

$$P_p(\text{percolation in } C_\infty) \geqslant P_p(\text{percolation in } C_\infty \cap B) = P_p(\text{percolation in } C'_\infty) \tag{3.1}$$

and hence that $p_c(M,d) \leqslant p_c(M,d-1) < 1$ for all $d \geqslant 3$.

Generalising the earlier work of Chayes and Chayes [7] in two dimensions, Falconer and Grimmett [23, 24] considered the behaviour of $p_c(M,d)$ for all $d \geqslant 2$ and large values of M. To state their conclusion, we digress briefly to recall the problem of site percolation in a lattice. Let $\mathbb{L}^d$ be the d-dimensional lattice with vertex set $\mathbb{Z}^d$ and edge set given by the adjacency relation: $\mathbf{x} \sim \mathbf{y}$ if and only if $|x_i - y_i| \leqslant 1$ for all i and $x_i = y_i$ for at least one value of i, where $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{y} = (y_1, \ldots, y_d)$. For $0 \leqslant p \leqslant 1$, we independently declare each vertex, or site, of $\mathbb{L}^d$ to be either open, with probability p, or else closed, with probability $1-p$; a set of open sites connected by edges is termed a cluster. We define *site percolation* to occur if an infinite cluster exists and let $p_c(\mathbb{L}^d)$ denote the critical probability for this process. Falconer and Grimmett proved that

$$p_c(M,d) \to p_c(\mathbb{L}^d) \quad \text{as } M \to \infty. \tag{3.2}$$

When $d = 2$, $\mathbb{L}^2$ is identical to the usual square lattice $\mathbb{Z}^2$; if $d \geqslant 3$, then $\mathbb{L}^d$ is obtained from the hypercubic lattice $\mathbb{Z}^d$ by an enhancement permitting extra connections between certain pairs of diagonally adjacent vertices, and so $p_c(\mathbb{L}^d) \leqslant p_c(\mathbb{Z}^d)$. In fact, this enhancement is 'essential' in the sense defined by Aizenman and Grimmett [1] (briefly, it has the capability for creating a doubly infinite path in the lattice where none existed previously) and so we have $p_c(\mathbb{L}^d) < p_c(\mathbb{Z}^d)$.

3.2 Disc percolation

Consider the fractal percolation process in the unit cube $[0,1]^3$ with $M \geqslant 2$ and retention probability $0 \leqslant p \leqslant 1$, as described above. We say that *disc percolation* occurs in the (x,y)-direction if the retained set C_∞ contains a set D such that there exists a homeomorphism $f\colon \Delta \to D$ and the following boundary conditions are satisfied (see Figure 3.1):

$$\begin{aligned}
(i) \quad f(\{0\} \times I) &\subseteq \{0\} \times I \times I \\
(ii) \quad f(\{1\} \times I) &\subseteq \{1\} \times I \times I \\
(iii) \quad f(I \times \{0\}) &\subseteq I \times \{0\} \times I \\
(iv) \quad f(I \times \{1\}) &\subseteq I \times \{1\} \times I.
\end{aligned} \tag{3.3}$$

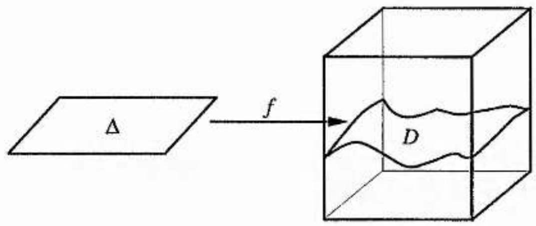


Figure 3.1: Disc percolation

Thus we think of disc percolation occurring if C_∞ contains a homeomorphic image of the unit disc spanning $[0,1]^3$ in the x- and y-directions. We make corresponding definitions for disc percolation in the (x, z)- and (y, z)-directions.

Observe that if disc percolation occurs in the (x, y)-direction then every continuous path $\gamma: I \to [0,1]^3$ such that $\gamma(0) \in I^2 \times \{0\}$ and $\gamma(1) \in I^2 \times \{1\}$ must have non-empty intersection with D.

Let $\theta_d(p) = P_p(\text{disc percolation})$ and define $p_d = \inf\{p: \theta_d(p) > 0\}$.

THEOREM 3.1: For all $M \geqslant 2$, we have $p_d < 1$.

Proof: (sketch) For $M \geqslant 5$, this is proved in essentially the same way as the original proof of the existence of the percolation phase in two dimensions; see Theorem 1.3 or Chayes *et al.* [8]. For $n \geqslant 0$, we declare a level-n cube to be *1-full* if at least $M^3 - 1$ of its M^3 level-$(n+1)$ subcubes are retained. We say that a level-n cube is *2-full* if at least $M^3 - 1$ of its subcubes are 1-full, and, inductively, that a level-n cube is *k-full* if at least $M^3 - 1$ of its subcubes are $(k-1)$-full. Observe then that for $M \geqslant 5$, any 2×2 block of four adjacent 1-full cubes not only forms a connected unit, but also contains a homeomorphic image of the unit disc spanning the block, as illustrated in Figure 3.2 (this can break down if $M < 5$).

Define a level-n cube to be *completely full* if it is k-full for all $k \geqslant 1$. By calculations similar to those of Chayes *et al.*, the probability that the level-0 cube $[0,1]^3$ is completely full is non-zero for values of p very close to 1. We shall show that if $[0,1]^3$ is completely full then disc percolation occurs, and hence deduce that $p_d < 1$.

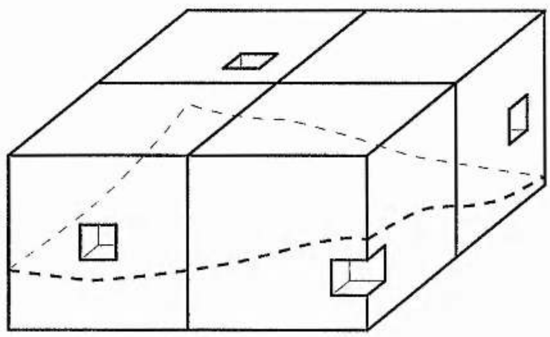


Figure 3.2: A block of four adjacent 1-full cubes contains an image of the disc

Suppose that $[0,1]^3$ is completely full. Let $f_0\colon \Delta \to C_0$ be the map given by $f_0(x,y) = (x,y,1/2)$ for all $(x,y) \in \Delta$ and let $D_0 = \mathrm{Im}(f) = \Delta \times \{1/2\}$. If any level-1 cube is either vacant or not completely full, we may smoothly deform D_0 so as to avoid that cube whilst still ensuring that the conditions of (3.3) are met; we obtain a new surface D_1. That is, there exists a map $f_1\colon \Delta \to C_1$ satisfying (3.3) with $\mathrm{Im}(f_1) = D_1$ and a homotopy $\tilde{f}\colon [0,1] \times \Delta \to C_0$ such that $\tilde{f}(0;x,y) = f_0(x,y)$ and $\tilde{f}(1;x,y) = f_1(x,y)$ for all $(x,y) \in \Delta$. In addition, we can easily ensure that $|f_1(x,y) - f_0(x,y)| \leqslant 1$ for all $(x,y) \in \Delta$.

Next we smoothly deform D_1, within the set C_1, so as to avoid any vacant or not completely full level-2 subcubes, obtaining a new surface D_2; this is possible since there is at most one such subcube per completely full level-1 cube and our condition that $M \geqslant 5$ guarantees that these subcubes cannot cluster together too much. Thus there is a map $f_2\colon \Delta \to C_2$ satisfying (3.3) with $\mathrm{Im}(f_2) = D_2$ and a homotopy $\tilde{f}\colon [1,2] \times \Delta \to C_1$ such that $\tilde{f}(1;x,y) = f_1(x,y)$ and $\tilde{f}(2;x,y) = f_2(x,y)$ for all $(x,y) \in \Delta$. In addition, since we do not have to deform D_1 too far so as to avoid the level-2 subcubes, we can ensure that $|f_2(x,y) - f_1(x,y)| \leqslant M^{-1}$ for all $(x,y) \in \Delta$.

Continuing in this fashion, we can extend the domain of $\tilde{f}$ to all of $\mathbb{R}^+ \times \Delta$ by piecing together the homotopies $\tilde{f}\colon [n, n+1] \times \Delta \to C_n$. We thus obtain a sequence $\{f_n\colon \Delta \to C_n\}_{n\in\mathbb{N}}$ of maps and a sequence $\{D_n\}_{n\in\mathbb{N}}$ of surfaces such that for all $n \in \mathbb{N}$, f_n satisfies the conditions of (3.3), $\mathrm{Im}(f_n) = D_n$, $\tilde{f}(n;x,y) = f_n(x,y)$ and $|f_n(x,y) - f_{n-1}(x,y)| \leqslant M^{-(n-1)}$ for all $(x,y) \in \Delta$. We conclude that as $n \to \infty$, f_n converges uniformly to a map $f\colon \Delta \to C_\infty$ satisfying (3.3) and with $\mathrm{Im}(f)$ homeomorphic to Δ, as required for disc percolation.

Finally, the cases $M = 4, 3, 2$ are dealt with by appealing to the results for $M = 16, 9, 4$ respectively, using the same method as in Theorem 1.4 and observing that $\theta_d(p; M) \geqslant \theta_d(q; M^2)$ when $p^{M^3+1} \geqslant 1 - (1-q)^{M^3}$; hence we have $p_d(M) < 1$ whenever $p_d(M^2) < 1$.

■

Proposition 3.2: Suppose that C_∞ and C'_∞ are realisations of fractal percolation in which disc percolation occurs in the (x, y)- and (x, z)-directions respectively. Then $C_\infty \cap C'_\infty$ contains a connected component intersecting both $\{0\} \times I^2$ and $\{1\} \times I^2$; that is, percolation occurs in $C_\infty \cap C'_\infty$.

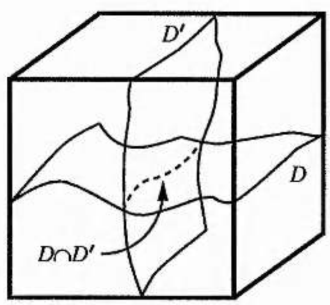


Figure 3.3: Intersecting disc percolation events

Proof: Let D, D' be the subsets of C_∞, C'_∞ respectively given by the definition of disc percolation and let $f : \Delta \to D$ be the homeomorphism satisfying the conditions of (3.3). Define $E = f^{-1}(D \cap D') \subseteq \Delta$. We shall show that E contains a connected component G intersecting both $L = \{0\} \times I$ and $R = \{1\} \times I$; the required component of $C_\infty \cap C'_\infty$ is then the one containing $f(G)$.

For $x \in \Delta$, let E_x denote the connected component of E containing x, or let $E_x = \varnothing$ if $x \notin E$. In addition, define $E_L = \bigcup\{E_x : x \in L\}$, $E_R = \bigcup\{E_x : x \in R\}$ and $E_C = \bigcup\{E_x : x \notin E_L \cup E_R\}$. Observe that all of the sets E_L, E_R and E_C are compact, since each may be written as the homeomorphic image of a decreasing limit of compact sets. We deduce that there exists $r > 0$ and a finite set of points $J \subseteq E_L$ such that $U_L = \bigcup_{x \in J} B(x, r)$ contains E_L, where $B(x, r)$ denotes the open ball of radius r with centre x. Likewise, we can find corresponding open sets $U_R \supseteq E_R$ and $U_C \supseteq E_C$; we further define $U = U_L \cup U_R \cup U_C$, and observe that the boundary ∂U of U consists entirely of circular arcs.

Suppose now that E contains no connected component intersecting both L and R, *i.e.* $E_L \cap E_R = \varnothing$. Then E_L, E_R and E_C are pairwise disjoint and so without loss of generality we may assume that U_L, U_R and U_C are pairwise disjoint. We may also assume that $U_L \cap R = \varnothing$, $U_R \cap L = \varnothing$, $U_C \cap (L \cup R) = \varnothing$, $U \not\supseteq B = I \times \{0\}$ and $U \not\supseteq T = I \times \{1\}$. Under the above conditions, we claim that there exists a continuous path $\gamma: I \to \Delta$ such that $\gamma(0) \in B$, $\gamma(1) \in T$ and $\gamma(t) \notin E$ for all $0 \leqslant t \leqslant 1$. To prove this, we start by choosing a path γ satisfying $\gamma(0) \in B \backslash U$ and $\gamma(1) \in T \backslash U$; we shall modify γ to avoid each of the components of E.

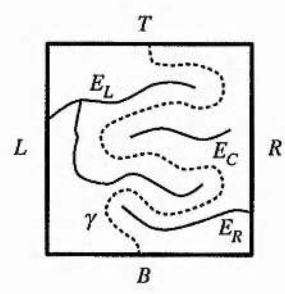


Figure 3.4: Modifying the path γ to avoid E

Suppose that $\gamma(t) \in E$ for some $0 < t < 1$ and let U_t denote the component of U containing $\gamma(t)$. Let $s_1 = \inf\{s: \gamma(s) \in U_t\}$ and $s_2 = \sup\{s: \gamma(s) \in U_t\}$; then $0 < s_1 < t < s_2 < 1$ and $\gamma(s_1), \gamma(s_2)$ both lie on ∂U_t. Now $\gamma(s_1)$ is in the same component, say Y_1, of $(\Delta \backslash U_t) \cup (B \cup T)$ as $\gamma(0)$, and $\gamma(s_2)$ is in the same component, say Y_2, as $\gamma(1)$; but we have $Y_1 = Y_2$ since U_t does not intersect both L and R. Hence $\gamma(s_1)$ and $\gamma(s_2)$ lie in the same component of $\partial U_t \cup (B \cup T)$, so we may remove the portion of path $\gamma\big((s_1, s_2)\big)$ and substitute in its place a portion of $\partial U_t \cup (B \cup T)$. Note that this new portion will have finite length since it is is composed of circular arcs of radius r and straight-line segments, so we can parameterise γ correctly. Note also that if γ intersects E on the lines B or T then we can simply alter the start or finish point for the path as appropriate.

We continue this modification process for each intersection of γ with E, observing that it must terminate in a finite number of steps since U consists only of finitely many components. We obtain a continuous path $\gamma: I \to \Delta$ such that $\gamma(0) \in B$, $\gamma(1) \in T$ and $\gamma(I) \cap E = \varnothing$. Mapping back to the unit cube, we see that $f \circ \gamma: I \to D$ is a continuous path satisfying $f \circ \gamma(0) \in I \times \{0\} \times I$, $f \circ \gamma(1) \in I \times \{1\} \times I$ and $f \circ \gamma(I) \cap D' = \varnothing$. Recall, however, our hypothesis that disc percolation in the (x, z)-direction occurs in C'_∞; this implies that every continuous path from $I \times \{0\} \times I$ to $I \times \{1\} \times I$ must intersect D' — a contradiction. We conclude that E contains a component intersecting both L and R, as required.

■

THEOREM 3.3: For all $M \geqslant 2$, we have $p_c \leqslant p_d^2 < p_d$; in particular, the critical probabilities for percolation and disc percolation are distinct.

Proof: Let $\varepsilon > 0$ and suppose that C_∞, C'_∞ are two independent realisations of fractal percolation in $[0,1]^3$ performed with retention probability $p = p_d + \varepsilon$. Let $A = \{$disc percolation in (x, y)-direction in $C_\infty\}$ and $A' = \{$disc percolation in (x, z)-direction in $C'_\infty\}$; then we have $P_p(A) = P_p(A') = \alpha > 0$. If A and A' occur simultaneously, then we have percolation occurring in $C_\infty \cap C'_\infty$ by Proposition 3.2; since A and A' are independent, we have $P_p(A \cap A') = \alpha^2$.

Finally observe that the set $C_\infty \cap C'_\infty$ is probabilistically equivalent to a single realisation of fractal percolation performed with retention probability p^2, and so $\theta((p_d + \varepsilon)^2) \geqslant \alpha^2 > 0$; since this holds for all $\varepsilon > 0$ we deduce that $p_c \leqslant p_d^2$.

■

3.2.1 Note on percolation in higher dimensions

When $d \geqslant 4$, it is possible to define various percolation events analogous to the concept of disc percolation considered above.

Let C_∞ be the random Cantor set given by the fractal percolation process in $[0,1]^d$ with $M \geqslant 2$ and $0 \leqslant p \leqslant 1$. For $1 \leqslant k \leqslant d-1$, we say that *k-ball percolation* occurs if there exists a set $D \subseteq C_\infty$ and a homeomorphism $f: I^k \to D$ such that the following boundary conditions are satisfied:

$$
\begin{aligned}
f(\{0\} \times I^{k-1}) &\subseteq \{0\} \times I^{k-1} \times I^{d-k} \\
f(\{1\} \times I^{k-1}) &\subseteq \{1\} \times I^{k-1} \times I^{d-k} \\
f(I \times \{0\} \times I^{k-2}) &\subseteq I \times \{0\} \times I^{k-2} \times I^{d-k} \\
f(I \times \{1\} \times I^{k-2}) &\subseteq I \times \{1\} \times I^{k-2} \times I^{d-k} \\
&\vdots \\
f(I^{k-1} \times \{1\}) &\subseteq I^{k-1} \times \{1\} \times I^{d-k}.
\end{aligned} \tag{3.4}
$$

Thus we think of k-ball percolation occurring if C_∞ contains a homeomorphic image of the unit ball of $\mathbb{R}^k$ spanning $[0,1]^d$ in the 1-,2-,..., k-directions. Note that the definition of 2-ball percolation in $[0,1]^3$ is identical to that of disc percolation, whilst 1-ball percolation is equivalent to arc-percolation.

We can now define critical probabilities $p_1, p_2, \ldots, p_{d-1}$ corresponding to the phase transitions to each of these percolation events. Clearly we have

$$p_1 \leqslant p_2 \leqslant \ldots \leqslant p_{d-1} \leqslant 1; \tag{3.5}$$

we would like to establish that this chain of inequalities is strict. By a method analogous to Theorem 3.1, we can prove that $p_{d-1} < 1$ for all $M \geqslant 2$. A version of Theorem 3.3 also exists: Let C_∞, C'_∞ be two independent realisations of fractal percolation in $[0,1]^d$ with retention probabilities $p > p_k$, $p' > p_{k'}$ respectively, where $1 \leqslant k, k' \leqslant d-1$. Let $A = \{k\text{-ball percolation in } C_\infty\}$ and $A' = \{k'\text{-ball percolation in } C'_\infty\}$; then we see that $P_{pp'}(A \cap A') > 0$.

Unfortunately, there appear to be considerable technical difficulties involved in generalising the kind of intersection result seen in Proposition 3.2. It seems reasonable, following the rule that 'co-dimensions add', to believe that intersecting k-ball and (rotated) k'-ball percolation events should produce $(k+k'-d)$-ball percolation, whence it would follow that $p_{k+k'-d} \leqslant p_k p_{k'}$. The best hope for making this argument rigorous is in the case $k' = d-1$, since this simplifies the geometrical problems somewhat; in fact, this case is all that would be needed to establish strict inequality in (3.5).

Other definitions of higher-dimensional percolation events are possible, and perhaps preferable for some purposes. As examples, for $1 \leqslant k \leqslant d-1$, we could define k-percolation to occur if the limit set C_∞ contains a homeomorphic image of a k-manifold, if every pre-fractal set C_n contains a bi-Lipschitz image of $[0,1]^k$, or a local definition where we only require small components homeomorphic to the unit ball of $\mathbb{R}^k$. However, technical difficulties obtaining intersection results with each of these definitions appear to remain.

3.3 Sheet percolation

As before, let $d \geqslant 2$, $M \geqslant 2$ and $0 \leqslant p \leqslant 1$ and let C_∞ denote the random Cantor set obtained by the fractal percolation process described in Section 3.1. We say that *sheet percolation* occurs in C_∞ (or in one of the pre-fractal sets C_n) if C_∞ (respectively, C_n) contains a 'surface' separating the left face L and the right face R of the cube $[0,1]^d$. To circumvent the topological difficulties in defining precisely what is meant by a surface, we shall work with the complementary set $C_\infty^{\mathrm{c}} = [0,1]^d \backslash C_\infty$ and define sheet percolation to occur if and only if C_∞^{c} does *not* contain a continuous path $\gamma\colon [0,1] \to C_\infty^{\mathrm{c}}$ such that $\gamma(0) \in L$ and $\gamma(1) \in R$. Note that when $d = 2$, this definition of sheet percolation corresponds precisely to the usual definition of percolation by connected components.

We let $\theta_s(p) = P_p(\text{sheet percolation in } C_\infty)$ and define the critical probability $p_s = p_s(M,d) = \inf\{p\colon \theta_s(p) > 0\}$. It is easy to see that $\theta_s(p) \leqslant \theta(p)$, since any surface spanning $[0,1]^d$ contains a connected component intersecting opposite faces of the cube, and hence $p_s \geqslant p_c$. As observed by Chayes *et al.* [9] in the case $d = 3$, it is easy to show that $p_s < 1$ by a method analogous to the original proof in the two-dimensional case. Moreover, when $d = 3$, note that $\theta_s(p) \geqslant \theta_d(p)$ and hence $p_s \leqslant p_d$.

Recall the definitions of the lattices $\mathbb{Z}^d$ and $\mathbb{L}^d$ from Section 3.1. We shall now define a new d-dimensional lattice. Let $\mathbb{M}^d$ be the lattice with vertex set $\mathbb{Z}^d$ and edge set given by the adjacency relation: $\mathbf{x} \sim \mathbf{y}$ if and only if $|x_i - y_i| \leqslant 1$ for all i. Thus $\mathbb{M}^d$ contains both $\mathbb{Z}^d$ and $\mathbb{L}^d$ as strict sublattices, and is obtained from $\mathbb{Z}^d$ by an enhancement permitting connections between *all* pairs of diagonally adjacent vertices. In addition, we define the sublattice $B_N(\mathbb{M}^d)$ of $\mathbb{M}^d$ of size $N \times \cdots \times N$ to be the lattice with vertex set $\{0,1,\ldots,N-1\}^d$ and edges inherited from $\mathbb{M}^d$. We shall consider the problem of site percolation on the lattice $\mathbb{M}^d$;

let $p_c(\mathbb{M}^d)$ denote the critical probability for this process.

Our main results in this section relate (the complement of) sheet percolation in the random Cantor set to site percolation on the lattice $\mathbb{M}^d$; in particular we shall show that as M increases, the critical probability for sheet percolation approaches $1 - p_c(\mathbb{M}^d)$.

THEOREM 3.4: For all $d \geqslant 2$ and $M \geqslant 2$, we have $p_s(M,d) \geqslant 1 - p_c(\mathbb{M}^d)$.

THEOREM 3.5: Let $d \geqslant 2$ and $p > 1 - p_c(\mathbb{M}^d)$. Then

$$P_p(\text{sheet percolation in } C_\infty^{[M]}) \to 1 \quad \text{as } M \to \infty.$$

THEOREM 3.6: For all $d \geqslant 2$,

$$p_s(M,d) \to 1 - p_c(\mathbb{M}^d) \quad \text{as } M \to \infty.$$

Proof of Theorem 3.6:

This is immediate from Theorems 3.4 and 3.5.

■

The reader should contrast Theorem 3.6 with (3.2). The lattice $\mathbb{M}^d$, rather than $\mathbb{L}^d$, appears because it is the existence of paths in the complement that determines whether or not sheet percolation occurs; for this, it is sufficient to have a sequence of vacant cubes between L and R meeting only at single points.

COROLLARY 3.7: For all $d \geqslant 3$, we have $p_c(M,d) < p_s(M,d)$ for all sufficiently large values of M.

Proof:

Combining (3.2) and Theorem 3.6, it is sufficient to show that

$$p_c(\mathbb{L}^d) < 1 - p_c(\mathbb{M}^d). \tag{3.6}$$

We note that $\mathbb{M}^d$ is obtained from $\mathbb{L}^d$ by an enhancement permitting extra connections between vertices, so certainly we have $p_c(\mathbb{M}^d) \leqslant p_c(\mathbb{L}^d)$. Similarly we have $p_c(\mathbb{L}^d) \leqslant p_c(\mathbb{Z}^d) < 1/2$ (where the last inequality is from Campanino and Russo [6]) which is sufficient for (3.6).

∎

Corollary 3.7 strengthens the result of Chayes *et al.* [9]. They proved that the inequality $p_c < p_s$ holds in three dimensions, working for technical reasons not in the unit cube but in the cuboid $[0,2] \times [0,2] \times [0,1]$, and again for sufficiently large values of M. We have extended the result to all $d \geqslant 3$ and removed the restriction on the geometry, although the requirement for large M remains.

Note also that when we apply Theorem 3.6 in the case $d = 2$, the concepts of percolation and sheet percolation are identical (subject to interchanging the axes), and hence we deduce that $p_c(M, 2) \to 1 - p_c(\mathbb{M}^2)$ as $M \to \infty$. In conjunction with (1.24), this shows that $p_c(\mathbb{M}^2) + p_c(\mathbb{Z}^2) = 1$, an equality observed by Sykes and Essam [55] and subsequently rigorously proved by Russo [52] and Kesten [34].

Exact values for critical probabilities of site percolation in these lattices are not known. The best known bounds for $p_c(\mathbb{Z}^2)$ are currently $0.556 < p_c(\mathbb{Z}^2) < 0.682$, the first inequality due to van den Berg and Ermakov [56], the second due to Zuev [61], with the exact value likely to be around 0.593.

We prove Theorem 3.4 in Section 3.3.1 and Theorem 3.5 in Section 3.3.2.

3.3.1 Lower bound on p_s

To prove that $p_s(M, d) \geqslant 1 - p_c(\mathbb{M}^d)$ for $d \geqslant 2$ and $M \geqslant 2$, we show that if $p < 1 - p_c(\mathbb{M}^d)$ then sheet percolation does not occur in C_∞, almost surely. We do this by comparing the pattern of cubes present in the level-n pre-fractal set C_n to the process of site percolation on a sublattice of $\mathbb{M}^d$, where each site is open with probability $q = 1 - p$. When this process is supercritical, there exist open paths crossing large boxes, and these are shown to preclude (almost surely) sheet percolation in C_∞.

Note that from the compactness of C_∞ it follows that

$$\{\text{sheet percolation in } C_\infty\} = \bigcap_{n=0}^{\infty} \{\text{sheet percolation in } C_n\}, \tag{3.7}$$

which is an intersection of a decreasing sequence of events, and therefore

$$P_p(\text{sheet percolation in } C_\infty) = \lim_{n\to\infty} P_p(\text{sheet percolation in } C_n). \quad (3.8)$$

We define another, stronger concept of percolation as follows: We say that *full sheet percolation* occurs in a set $S \subseteq [0,1]^d$ if the interior of S separates L and R, *i.e.* if and only if S^*, defined by $S^* = \overline{[0,1]^d \setminus S}$, does not contain a continuous path $\gamma: [0,1] \to S^*$ such that $\gamma(0) \in L$ and $\gamma(1) \in R$. Thus we may think of a family S of level-n cubes, forming a surface separating L and R, as being full if all the pairs of adjacent cubes $\{C', C''\}$ that are necessary to block paths in the complement satisfy $\dim(C' \cap C'') = d-1$, that is, C' and C'' intersect in a $(d-1)$-dimensional 'face', rather than an 'edge' of dimension less than $(d-1)$.

LEMMA 3.8:

$$P_p(\text{sheet percolation in } C_\infty) = \lim_{n\to\infty} P_p(\text{full sheet percolation in } C_n).$$

Proof: This proof is based upon that of Lemma 5 of Falconer and Grimmett [24], with the necessary generalisations to $d \geqslant 3$. Clearly

$$\{\text{full sheet percolation in } C_n\} \subseteq \{\text{sheet percolation in } C_n\} \quad (3.9)$$

and so

$$\begin{aligned} \bigcap_{n=0}^{\infty} \{\text{full sheet percolation in } C_n\} &\subseteq \bigcap_{n=0}^{\infty} \{\text{sheet percolation in } C_n\} \\ &= \{\text{sheet percolation in } C_\infty\} \quad (3.10) \end{aligned}$$

by (3.7). Now $\{\text{full sheet percolation in } C_n\}_{n\geqslant 0}$ is a decreasing sequence of events, and therefore it will be sufficient to show that for all $n \geqslant 1$,

$$P_p\big(\{\text{sheet percolation in } C_\infty\} \cap \{\text{no full sheet percolation in } C_n\}\big) = 0. \quad (3.11)$$

Fix $n \geqslant 1$. Write C_n as the union of a set of level-n cubes $\{C(1), \ldots, C(r)\}$. Now for $0 \leqslant l \leqslant d$, let

$$\tilde{D}_l = \bigcup_{1\leqslant i<j\leqslant r} \{C(i) \cap C(j) : \dim\big(C(i) \cap C(j)\big) = l\}$$

and for $0 \leqslant k \leqslant d$, let

$$D_k = \bigcup_{l<k} \tilde{D}_l \setminus \bigcup_{l \geqslant k} \tilde{D}_l;$$

note that $\varnothing = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_d$. We think of D_k as the set of cube intersections that have dimension at most $k-1$ and that are not contained in a cube intersection having dimension greater than $k-1$.

Define *k-percolation* to occur in C_n if sheet percolation occurs in $C_n \backslash D_k$. Observe then that 0-percolation and sheet percolation in C_n are equivalent concepts, as are $(d-1)$-percolation and full sheet percolation. Let A_n^k denote the event $\{k\text{-percolation occurs in } C_n\}$ and note that $A_n^0 \supseteq A_n^1 \supseteq \cdots \supseteq A_n^{d-1}$.

Suppose that for some $k < d-1$, we have k-percolation but no $(k+1)$-percolation in C_n. Then $D_{k+1} \backslash D_k$ is a non-empty union of k-dimensional edges $\{F_i\}$ and there is a minimal non-empty subset G of $\{F_i\}$, say $G = \{F_i \colon 1 \leqslant i \leqslant s\}$, such that sheet percolation does not occur in $C_n \backslash (D_k \cup G)$. We may then choose a path $\gamma \colon [0,1] \to C_n^c \cup (D_k \cup G)$ such that $\gamma(0) \in L$ and $\gamma(1) \in R$. Since G was taken to be minimal, this path must intersect every $F_i \in G$; without loss of generality, we may assume that γ intersects each F_i exactly once (otherwise we can easily modify γ to do so).

Fix $1 \leqslant i \leqslant s$ and let $C_i[1]$ and $C_i[2]$ denote the two (vacant) level-n cubes that γ passes through on either side of its intersection with F_i. There are $2^{d-k}-2$ other level-n cubes (or notional cubes, if lying outside $[0,1]^d$) that contain F_i; we shall label these as $C_i[3], \ldots, C_i[2^{d-k}]$. For every $m \geqslant n$, each one of these level-n cubes contains $M^{k(m-n)}$ level-m subcubes that intersect F_i. We choose a subset S_i^m of these level-m subcubes, one from each of the $C_i[j]$, $j = 3, \ldots, 2^{d-k}$, such that $\bigcap S_i^m$ is a k-dimensional subset of F_i; note that there are $M^{k(m-n)}$ possible such choices of a set S_i^m.

If the set S_i^m of level-m cubes were to be removed from $C_n \backslash D_k$ then the interiors of the two vacant cubes $C_i[1]$ and $C_i[2]$ would be joined together by a small 'crack' in F_i, as shown in Figure 3.5. Moreover, if this were to be repeated for each $F_i \in G$ then the resulting set $C_n \backslash (D_k \cup \bigcup_{1 \leqslant i \leqslant s} S_i^m)$ would allow a vacant path, actually a slight modification of γ, to cross the cube intersecting both L and R — that is, sheet percolation would be prevented.

We thus wish to investigate the probability that such an event occurs in C_m, given a set C_n satisfying $A_n^k \backslash A_n^{k+1}$. Let $1 \leqslant i \leqslant s$ and $m \geqslant n$, define $C_i[j]$,

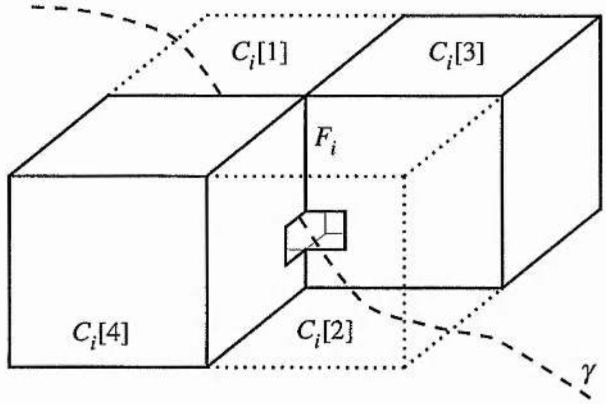


Figure 3.5: A 'crack' γ joining the vacant cubes $C_i[1]$ and $C_i[2]$ via F_i

$j = 1, \ldots, 2^{d-k}$, as above and choose a suitable set S_i^m of level-m cubes; observe that each of these level-m cubes survives in C_m independently with probability p^{m-n}. Let

$$E_i^m = \{C_m \text{ admits a vacant path joining the interiors of } C_i[1] \text{ and } C_i[2]\}.$$

Then $E_i^m \supseteq \{\operatorname{int}(S_i^m \cap C_m) = \varnothing\}$, and hence it is easily seen that

$$P_p(E_i^m) \geqslant (1 - p^{m-n})^{2^{d-k}-2}. \tag{3.12}$$

Define $Q^m = \operatorname{card}\{i : E_i^m \text{ does not occur}\}$; from the above comments it follows that if sheet percolation occurs in C_m then Q^m is non-zero. By (3.12) we have

$$\mathbf{E}(Q^m | C_n) = s\big(1 - P_p(E_1^m)\big) \leqslant s\big(1 - (1 - p^{m-n})^{2^{d-k}-2}\big) \tag{3.13}$$

where $\mathbf{E}$ denotes expectation, and hence

$$\begin{aligned} P_p(\text{sheet percolation in } C_m \,|\, C_n) &\leqslant P_p(Q^m \geqslant 1 | C_n) \\ &\leqslant \mathbf{E}(Q^m | C_n) \leqslant s\big(1 - (1 - p^{m-n})^{2^{d-k}-2}\big) \\ &\to 0 \quad \text{as } m \to \infty. \end{aligned} \tag{3.14}$$

The limit in (3.14) holds for all realisations C_n in which $A_n^k \backslash A_n^{k+1}$ occurs; therefore

$$\lim_{m\to\infty} P_p\big(\text{sheet percolation in } C_m \;\big|\; A_n^k \backslash A_n^{k+1}\big) = 0. \tag{3.15}$$

For every $n \geqslant 1$, we have

$$\begin{aligned} &\{\text{sheet percolation in } C_\infty\} \cap \{\text{no full sheet percolation in } C_n\} \\ &= \lim_{m\to\infty} \{\text{sheet percolation in } C_m\} \cap (A_n^0 \backslash A_n^{d-1}) \end{aligned} \tag{3.16}$$

since {sheet percolation in C_m} $\subseteq A_n^0$ for all $m \geqslant n$. Also observe that

$$A_n^0 \backslash A_n^{d-1} = \bigcup_{k=0}^{d-2} A_n^k \backslash A_n^{k+1} \tag{3.17}$$

and hence

$$\begin{aligned} & P_p\big(\{\text{sheet percolation in } C_\infty\} \cap \{\text{no full sheet percolation in } C_n\}\big) \\ & = \sum_{k=0}^{d-2} \lim_{m\to\infty} P_p\big(\{\text{sheet percolation in } C_m\} \cap (A_n^k \backslash A_n^{k+1})\big) \\ & = 0 \end{aligned} \tag{3.18}$$

by (3.15), giving (3.11) as required to complete the proof.

■

Proof of Theorem 3.4:

Let $p < 1 - p_c(\mathbb{M}^d)$ and write $q = 1 - p$. We consider site percolation on the lattice $\mathbb{M}^d$, with sites being declared open independently at random with probability q. Let $\theta_q\big(B_N(\mathbb{M}^d)\big)$ denote the probability that there exists an open cluster intersecting both the left face $\{0\} \times \{0, \ldots, N-1\}^{d-1}$ and the right face $\{1\} \times \{0, \ldots, N-1\}^{d-1}$ of the box $\{0, \ldots, N-1\}^d$. Since q is greater than the critical probability for site percolation in $\mathbb{M}^d$, it follows from Theorem (6.125) of Grimmett [29] that there exists $\tau > 0$ such that

$$\theta_q\big(B_N(\mathbb{M}^d)\big) \geqslant \tau \tag{3.19}$$

for all $N > 0$.

For each $n \geqslant 1$, we define $C_n^* = \overline{[0,1]^d \backslash C_n}$, giving an increasing sequence $C_0^* \subseteq C_1^* \subseteq \cdots$ of closed sets, and note that full sheet percolation occurs in C_n if and only if C_n^* does not contain a continuous path $\gamma : [0,1] \to C_n^*$ such that $\gamma(0) \in L$ and $\gamma(1) \in R$.

In order to obtain estimates on P_p(full sheet percolation in C_n), we shall set up a correspondence between the sets C_n^* and site percolation on $B_{M^n}(\mathbb{M}^d)$. After rescaling by a factor of M^n, the level-n cubes correspond to the sites of $B_{M^n}(\mathbb{M}^d)$, with two sites being considered adjacent if and only if the corresponding cubes have at least a point in common; the cubes present in C_n^* shall correspond to the open vertices in $B_{M^n}(\mathbb{M}^d)$. By this comparison, conditioning on full retention at level $(n-1)$, we see that

$$P_p(\text{no full sheet percolation in } C_n \mid C^*_{n-1} = \varnothing) = \theta_q\big(B_{M^n}(\mathbb{M}^d)\big) \geqslant \tau. \quad (3.20)$$

Therefore

$$P_p(\text{full sheet percolation in } C_n) \leqslant \prod_{j=1}^{n}\Big(1 - \theta_q\big(B_{M^j}(\mathbb{M}^d)\big)\Big) \leqslant (1-\tau)^n \quad (3.21)$$

for each $n \geqslant 1$ by (3.20). Since $\tau > 0$, applying Lemma 3.8 we find that

$$P_p(\text{sheet percolation in } C_\infty) = 0 \quad (3.22)$$

as required.

■

3.3.2 Upper bound on p_s

The proof of Theorem 3.5 is long, but can be divided into several parts. Let $d \geqslant 2$, $p > 1 - p_c(\mathbb{M}^d)$, and choose an $\varepsilon > 0$ such that $(1-\varepsilon)p > 1 - p_c(\mathbb{M}^d)$. Firstly, for $n \geqslant 1$ and $0 \leqslant m \leqslant n$, we define the notions of goodness and availability on the level-m cubes; roughly, a level-m cube is good if it contains sufficiently many interlocking shells of smaller cubes. We then use geometrical arguments to show that if the level-0 cube $[0,1]^d$ is good then we have sheet percolation in C_n. Next, probabilistic arguments are used to show that this event occurs with probability close to 1 providing that M is sufficiently large; that is, there exists $M(\varepsilon)$ such that for all $M \geqslant M(\varepsilon)$, we have

$$P_p(\text{sheet percolation in } C_n) \geqslant 1 - \varepsilon \quad (3.23)$$

for all $n \geqslant 1$. We deduce using (3.8) and letting $\varepsilon \to 0$ that

$$P_p(\text{sheet percolation in } C_\infty^{[M]}) \to 1 \quad \text{as } M \to \infty \quad (3.24)$$

and hence $p_s(M,d) \leqslant p$ for all sufficiently large M.

We shall assume that M is divisible by 5, although it will be clear that the method of the proof works for any $M \geqslant 5$, with the necessary slight modifications if M is not divisible by 5.

We adopt the following notation for labelling subcubes of $[0,1]^d$. Let $J = (\mathbb{Z}_M)^d = \{0, 1, \ldots, M-1\}^d$ and, for $n \geqslant 1$, $\Sigma_n = J^n$; let $\Sigma_0 = \{\varnothing\}$. Thus Σ_n

corresponds to the set of level-n subcubes of $[0,1]^d$; we denote elements of Σ_n by their *index*

$$\mathbf{I}^{(n)} = (\mathbf{i}_1, \dots, \mathbf{i}_n) = ((i_{1,1}, \dots, i_{1,d}), \dots, (i_{n,1}, \dots, i_{n,d}))$$

where each $\mathbf{i}_j \in J$, $1 \leqslant j \leqslant n$. With each index $\mathbf{I}^{(n)}$ we associate the level-n cube $C[\mathbf{I}^{(n)}]$ given by

$$C[\mathbf{I}^{(n)}] = c[\mathbf{I}^{(n)}] + [0, M^{-n}]^d$$

where

$$c[\mathbf{I}^{(n)}] = \left(\sum_{j=1}^{n} i_{j,1} M^{-j}, \dots, \sum_{j=1}^{n} i_{j,d} M^{-j} \right);$$

finally let $C[\varnothing]$ denote the level-0 cube $[0,1]^d$.

Let $\Sigma = \bigcup_{n \geqslant 0} \Sigma_n$ and suppose that we are given a map $\omega: \Sigma \to \{0,1\}$. For each $\mathbf{I}^{(n)} = (\mathbf{i}_1, \dots, \mathbf{i}_n) \in \Sigma_n$, $n \geqslant 1$, we define the indicator function $1_\omega[\mathbf{I}]$ by

$$1_\omega[\mathbf{I}] = \prod_{j=1}^{n} \omega\big[(\mathbf{i}_1, \dots, \mathbf{i}_j)\big]$$

and observe that

$$1_\omega[\mathbf{I}^{(n+1)}] = 1_\omega[\mathbf{I}^{(n)}] \omega[\mathbf{I}^{(n+1)}] \tag{3.25}$$

for every $\mathbf{I}^{(n+1)} = (\mathbf{I}^{(n)}, \mathbf{i}_{n+1}) \in \Sigma_{n+1}$. By this construction, the set C_n is the union of those level-n cubes $C[\mathbf{I}^{(n)}]$ satisfying $1_\omega[\mathbf{I}^{(n)}] = 1$. Defining $\Omega = \{0,1\}^\Sigma$, we see that elements $\omega \in \Omega$ represent realisations of fractal percolation in the cube; the probability measure P_p on subsets of Ω is defined by (1.4).

Let $\mathbf{I}^{(n)} \in \Sigma_n$ and $\mathbf{k} = (k_1, \dots, k_d) \in \{0, \dots, 4\}^d$. We define the level-$n$ *block* $B[\mathbf{I}^{(n)}; \mathbf{k}]$ by

$$B[\mathbf{I}^{(n)}; \mathbf{k}] = c[\mathbf{I}^{(n)}] + \Big(\frac{1}{5} k_1 M^{-n}, \dots, \frac{1}{5} k_d M^{-n}\Big) + \Big[0, \frac{1}{5} M^{-n}\Big]^d.$$

Then each level-n cube $C[\mathbf{I}^{(n)}]$ can be written as the union of the 5^d level-n blocks contained therein, and each level-n block $B[\mathbf{I}^{(n)}; \mathbf{k}]$ is the union of $(M/5)^d$ level-$(n+1)$ subcubes of $C[\mathbf{I}^{(n)}]$.

Define the *annulus* $A[\mathbf{I}^{(n)}; \mathbf{k}]$ around a block $B[\mathbf{I}^{(n)}; \mathbf{k}]$ by

$$A[\mathbf{I}^{(n)}; \mathbf{k}] = \Big\{ c[\mathbf{I}^{(n)}] + \Big(\frac{1}{5} k_1 M^{-n}, \dots, \frac{1}{5} k_d M^{-n}\Big) + \Big[-\frac{1}{5} M^{-n}, \frac{2}{5} M^{-n}\Big]^d \Big\} \setminus \mathrm{int}\, B[\mathbf{I}^{(n)}; \mathbf{k}]$$

so that $A[\mathbf{I}^{(n)};\mathbf{k}]$ is composed of the 3^d-1 level-n blocks touching $B[\mathbf{I}^{(n)};\mathbf{k}]$ (or notional blocks outside $[0,1]^d$ if $B[\mathbf{I}^{(n)};\mathbf{k}]$ intersects the boundary of $[0,1]^d$). Note that, with our definitions, no extra difficulties will arise with those annuli not completely contained within $[0,1]^d$. In addition, we define $\partial^{(i)}A[\mathbf{I}^{(n)};\mathbf{k}]$ and $\partial^{(o)}A[\mathbf{I}^{(n)};\mathbf{k}]$ to be respectively the inner and outer components of the boundary of $A[\mathbf{I}^{(n)};\mathbf{k}]$.

Fix $n \geqslant 1$ and a map $\omega\colon \Sigma \to \{0,1\}$, representing a particular realisation of fractal percolation. For every $m \leqslant n$, we now define the notions of goodness and availability for each level-m cube $C[\mathbf{I}^{(m)}]$, $\mathbf{I}^{(m)} \in \Sigma_m$, by induction on $m = n, n-1, \ldots, 0$ as follows:

$m = n$: We declare all level-n cubes $C[\mathbf{I}^{(n)}]$, $\mathbf{I}^{(n)} \in \Sigma_n$, to be *good*, and declare $C[\mathbf{I}^{(n)}]$ to be ***available*** if $\omega[\mathbf{I}^{(n)}] = 1$.

$m < n$: Suppose that we have determined the availability of $C[\mathbf{I}]$ for every $\mathbf{I} \in \Sigma_{m+1} \cup \cdots \cup \Sigma_n$. Given subsets D, E and S of $\mathbb{R}^d$, we say that S contains a ***full sheet separating*** D ***and*** E if there is no continuous path $\gamma : [0,1] \to \overline{\mathbb{R}^d \backslash S}$ such that $\gamma(0) \in D$ and $\gamma(1) \in E$. We say that the block $B[\mathbf{I}^{(m)};\mathbf{k}]$ is ***isolated*** if the set

$$S = \bigcup \{C[\tilde{\mathbf{I}}^{(m+1)}] \colon C[\tilde{\mathbf{I}}^{(m+1)}] \text{ is available, } \tilde{\mathbf{I}}^{(m+1)} \in \Sigma_{m+1}\} \cup \{\mathbb{R}^d \backslash [0,1]^d\}$$

contains a full sheet separating $\partial^{(i)}A[\mathbf{I}^{(m)};\mathbf{k}]$ and $\partial^{(o)}A[\mathbf{I}^{(m)};\mathbf{k}]$. Figure 3.6 illustrates an isolated block when $d = 2$.

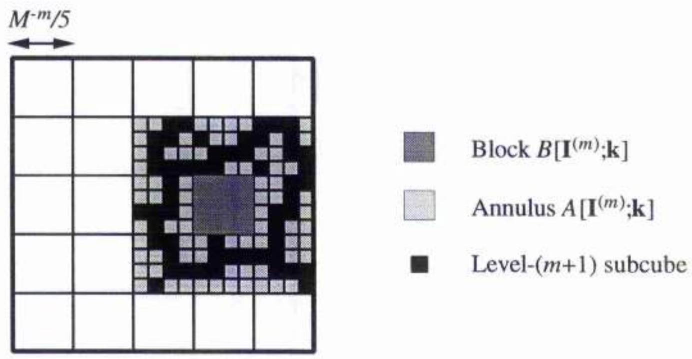


Figure 3.6: A level-m cube $C[\mathbf{I}^{(m)}]$ containing an isolated block $B[\mathbf{I}^{(m)};\mathbf{k}]$

For subsets X, Y of $\mathbb{R}^d$, we define $\operatorname{dist}(X,Y) = \inf\{d(\mathbf{x},\mathbf{y}) : \mathbf{x} \in X,\ \mathbf{y} \in Y\}$, where $\mathbf{x} = (x_1, \ldots, x_d)$, $\mathbf{y} = (y_1, \ldots, y_d)$, and $d(\mathbf{x}, \mathbf{y}) = \max_{1 \leqslant i \leqslant d} |x_i - y_i|$, with the convention that $\inf \varnothing = \infty$.

Let $\mathbf{I}^{(m)}(1) < \mathbf{I}^{(m)}(2) < \ldots < \mathbf{I}^{(m)}(M^{dm})$ be some fixed ordering of Σ_m. Using this ordering, we determine the goodness of each $C[\mathbf{I}^{(m)}(j)]$, $1 \leqslant j \leqslant M^{dm}$, in turn as follows: For each $1 \leqslant j \leqslant M^{dm}$, let

$$P(j) = \bigcup_{l<j} \{C[\mathbf{I}^{(m)}(l)] \colon C[\mathbf{I}^{(m)}(l)] \text{ is not good}\}$$

be the set of level-m cubes preceding $C[\mathbf{I}^{(m)}(j)]$ that have been examined and found to be not good. We declare the cube $C[\mathbf{I}^{(m)}(j)]$ to be *good* if $B[\mathbf{I}^{(m)}(j); \mathbf{k}]$ is isolated for every $\mathbf{k} \in \{0, \ldots, 4\}^d$ such that

$$\operatorname{dist}\big(B[\mathbf{I}^{(m)}(j); \mathbf{k}], P(j)\big) \geqslant 2M^{-m}/5.$$

In addition, we declare the cube $C[\mathbf{I}^{(m)}(j)]$ to be *available* if it is both good and $\omega[\mathbf{I}^{(m)}(j)] = 1$.

Informally, we have defined a level-m cube $C[\mathbf{I}^{(m)}]$ to be good if it contains a favourable arrangement of full sheets of smaller cubes (the exact arrangement required depending upon the status of the level-m cubes previously examined), and to be available if it is both good and retained for the next level of the inductive definition. Where convenient, we shall use the indicator function $1_A[\mathbf{I}^{(m)}]$, taking the value 1 if $C[\mathbf{I}^{(m)}]$ is available and 0 otherwise.

Using this procedure, we can determine the goodness of the level-0 cube $C[\varnothing] = [0, 1]^d$.

PROPOSITION 3.9: If $C[\varnothing]$ is good then sheet percolation occurs in C_n.

In order to prove Proposition 3.9, we require the following result. For a subset F_m of Σ_m, let $\bigcup(F_m)$ denote the union

$$\bigcup_{\mathbf{I}^{(m)} \in F_m} C[\mathbf{I}^{(m)}].$$

LEMMA 3.10: Let $m < n$ and let F_m be a subset of Σ_m satisfying $1_A[\mathbf{I}^{(m)}] = 1$ and $1_\omega[\mathbf{I}^{(m)}] = 1$ for all $\mathbf{I}^{(m)} \in F_m$ and such that $\bigcup(F_m)$ contains a full sheet separating L and R.

Then there exists $F_{m+1} \subseteq \Sigma_{m+1}$ such that $1_A[\mathbf{I}^{(m+1)}] = 1$ and $1_\omega[\mathbf{I}^{(m+1)}] = 1$ for all $\mathbf{I}^{(m+1)} \in F_{m+1}$ and $\bigcup(F_{m+1})$ contains a full sheet separating L and R.

Proof: Let $S_m = \bigcup(F_m)$ and define the *core* $\tilde{S}_m$ of S_m by

$$\tilde{S}_m = \bigcup_{\mathbf{I}^{(m+1)} \in \Sigma_{m+1}} \left\{ C[\mathbf{I}^{(m+1)}] \colon \operatorname{dist}\left(C[\mathbf{I}^{(m+1)}], [0,1]^d \backslash S_m\right) \geqslant \frac{2}{5} M^{-m} \right\}$$

so that $\tilde{S}_m$ is the union of those level-$(m+1)$ cubes which are distance at least $2M^{-m}/5$ from $[0,1]^d \backslash S_m$. We note that since S_m consists of cubes of side-length M^{-m}, its core $\tilde{S}_m$ must also contain a full sheet separating L and R.

Pick an $\mathbf{I}^{(m+1)} \in \Sigma_{m+1}$ such that $C[\mathbf{I}^{(m+1)}] \subseteq \tilde{S}_m$; then we have

$$C[\mathbf{I}^{(m+1)}] \subseteq B[\mathbf{I}^{(m)}; \mathbf{k}] \subseteq \left\{ A[\mathbf{I}^{(m)}; \mathbf{k}] \cup B[\mathbf{I}^{(m)}; \mathbf{k}] \right\} \subseteq S_m \tag{3.26}$$

for some $\mathbf{k} \in \{0, \ldots, 4\}^d$. Since $C[\mathbf{I}^{(m)}]$ consists of 5^d equal level-m blocks each of side-length $M^{-m}/5$, it is easy to see that we have

$$\operatorname{dist}\left(B[\mathbf{I}^{(m)}; \mathbf{k}], [0,1]^d \backslash S_m\right) \geqslant 2M^{-m}/5. \tag{3.27}$$

All the level-m cubes contained in S_m are good, and hence we deduce from the definition of goodness that the block $B[\mathbf{I}^{(m)}; \mathbf{k}]$ must be isolated.

We define

$$S_{m+1} = S_m \cap \bigcup_{\mathbf{I}^{(m+1)} \in \Sigma_{m+1}} \left\{ C[\mathbf{I}^{(m+1)}] \colon 1_A[\mathbf{I}^{(m+1)}] = 1 \right\}$$

and let F_{m+1} be the set of indices of the level-$(m+1)$ cubes contained in S_{m+1}, so that $S_{m+1} = \bigcup(F_{m+1})$. Pick $\mathbf{I}^{(m+1)} = (\mathbf{I}^{(m)}, \mathbf{i}_{m+1}) \in F^{m+1}$; we note that since $1_\omega[\mathbf{I}^{(m)}] = 1$ and $\omega[\mathbf{I}^{(m+1)}] = 1$ we have $1_\omega[\mathbf{I}^{(m+1)}] = 1$ by (3.25).

Suppose that S_{m+1} does not contain a full sheet separating L and R, that is, there exists a sequence $\Gamma = \{C(1), \ldots, C(r)\}$ of level-$(m+1)$ cubes such that

$$\begin{aligned}
C(j) &\not\subseteq S_{m+1} && \text{for all } 1 \leqslant j \leqslant r \\
C(1) \cap L &\neq \varnothing && \\
C(r) \cap R &\neq \varnothing && \\
C(j) \cap C(j+1) &\neq \varnothing && \text{for all } 1 \leqslant j < r.
\end{aligned} \tag{3.28}$$

Since $\tilde{S}_m$ does contain a full sheet separating L and R, we must have $C(i) \subseteq \tilde{S}_m$ for some $1 \leqslant i \leqslant r$; let $B[\mathbf{I}^{(m)}; \mathbf{k}]$ be the level-m block containing $C(i)$. By

(3.26), we have $A[\mathbf{I}^{(m)};\mathbf{k}] \subseteq S_m$, and hence we see that there is a sequence $\Gamma' = \{C(s),\ldots,C(t)\} \subseteq \Gamma$ of level-$(m+1)$ cubes such that

$$\begin{aligned} C(s)\cap\partial^{(i)}A[\mathbf{I}^{(m)};\mathbf{k}] &\neq \varnothing \\ C(t)\cap\partial^{(o)}A[\mathbf{I}^{(m)};\mathbf{k}] &\neq \varnothing \\ C(j)\cap C(j+1) &\neq \varnothing \qquad \text{for all } s\leqslant j<t \end{aligned} \tag{3.29}$$

and $C(j)$ is not available for any $s \leqslant j \leqslant t$. But this means that the block $B[\mathbf{I}^{(m)};\mathbf{k}]$ is not isolated, contradicting the above.

Hence we conclude that S_{m+1} does contain a full sheet separating L and R, as required.

■

Proof of Proposition 3.9:

Suppose that $C[\varnothing]$ is good; then for every $\mathbf{k} \in \{0,\ldots,4\}^d$, the block $B[\varnothing;\mathbf{k}]$ is isolated. We let

$$F_1 = \{\mathbf{I}^{(1)} \in \Sigma_1 \colon 1_A[\mathbf{I}^{(1)}] = 1\}$$

and so we see that the set $\bigcup(F_1)$ contains a full sheet separating L and R. We note that $1_\omega[\mathbf{I}^{(1)}] = \omega[\mathbf{I}^{(1)}] = 1$ for every $\mathbf{I}^{(1)} \in F_1$.

We now repeatedly apply Lemma 3.10 with $m = 1,2,\ldots,n-1$ to deduce that there exist sets $F_2, F_3, \ldots, F_n$ such that for every $1 \leqslant m \leqslant n$, we have $1_A[\mathbf{I}^{(m)}] = 1$ and $1_\omega[\mathbf{I}^{(m)}] = 1$ for all $\mathbf{I}^{(m)} \in F_m$ and $\bigcup(F_m)$ contains a full sheet separating L and R. In particular, when $m = n$, there exists a set $S = \bigcup(F_n)$ such that $1_\omega[\mathbf{I}^{(n)}] = 1$ for every $\mathbf{I}^{(n)} \in F_n$ and S contains a full sheet separating L and R. Therefore $S \subseteq C_n$ and sheet percolation occurs S and hence in C_n.

■

We now consider site percolation on the lattice $\mathbb{M}^d$, with sites being declared open independently at random with probability q, and let P_q denote the natural product probability measure. Let $\{0 \leftrightarrow \partial B(N)\}$ denote the event {there exists an open path from the origin to a vertex of $\partial B(N)$}, where

$$\partial B(N) = \{\mathbf{x} \in \mathbb{Z}^d : \max\{|x_i| : 1 \leqslant i \leqslant d\} = N\}$$

is the surface of the box of side-length $2N$ centred at the origin.

LEMMA 3.11: Suppose that $q < p_c(\mathbb{M}^d)$. Then there exists $\sigma_q > 0$ such that for all $N \geqslant 0$ we have

$$P_q\big(0 \leftrightarrow \partial B(N)\big) \leqslant \exp(-N\sigma_q).$$

Lemma 3.11 is a modified version of a result of Menshikov [43], restated as Theorem 3.4 of Grimmett [29]. There it is given in the case of bond percolation on $\mathbb{Z}^d$, but the proof adapts readily to site percolation on $\mathbb{M}^d$.

Lemma 3.11 can be used to estimate the probability that a level-m block $B[\mathbf{I}^{(m)}; \mathbf{k}]$ is isolated. Recall that $\varepsilon > 0$ satisfies $(1-\varepsilon)p > 1-p_c(\mathbb{M}^d)$ and define $\pi = (1-\varepsilon)p$. Let $m < n$ and suppose that each level-$(m+1)$ cube is available with probability π, independently of all other level-$(m+1)$ cubes; let $\overline{P_\pi}$ denote the corresponding product probability measure on $\{0,1\}^{\Sigma_{m+1}}$.

We compare C_{m+1} to site percolation on a sublattice of $\mathbb{M}^d$ as follows: After rescaling by a factor of M^{m+1}, the non-available cubes in C_{m+1} correspond to the open sites of $B_{M^{m+1}}$; two sites are adjacent if and only if the corresponding cubes have at least one point in common. Then it can be seen that the block $B[\mathbf{I}^{(m)}; \mathbf{k}]$ is not isolated only if there exists an open path from one of the sites corresponding to a level-$(m+1)$ subcube of $B[\mathbf{I}^{(m)}; \mathbf{k}]$ to the boundary of the box of side-length $2M/5$ centred at this site.

Since $1-\pi < p_c(\mathbb{M}^d)$, we may apply Lemma 3.11 to deduce that

$$\begin{aligned} \overline{P_\pi}(B[\mathbf{I}^{(m)}; \mathbf{k}] \text{ is not isolated}) &\leqslant (M/5)^d P_{1-\pi}(0 \leftrightarrow \partial B(M/5)) \\ &\leqslant (M/5)^d \exp(-M\sigma_{1-\pi}/5) \qquad (3.30) \end{aligned}$$

We shall take $M = M(\varepsilon)$ sufficiently large so that

$$M^d \exp(-M\sigma_{1-\pi}/5) < \varepsilon. \qquad (3.31)$$

Suppose that we are given a sequence $\{X_k\}_{k\geqslant 1}$ of random variables, each taking values in $\{0,1\}$, with corresponding state space $\Xi = \{0,1\}^{\mathbb{N}}$. For $0 \leqslant \rho \leqslant 1$, we let $\overline{P_\rho}$ denote the Bernoulli product probability measure on Ξ, so that $\overline{P_\rho}(X_k = 1) = \rho$ independently for all $k \geqslant 1$. We further define $\mathcal{F}_k$ to be the σ-algebra generated by $\{X_1, \ldots, X_k\}$.

LEMMA 3.12: For $l \geqslant 1$, let $\{X_k\}_{1\leqslant k\leqslant l}$ be a finite sequence of random variables, each taking values in $\{0,1\}$, and let P be a given probability measure on $\{0,1\}^l$. Suppose that there exists $0 \leqslant \rho \leqslant 1$ such that

$$P\big(X_k = 1\big|\mathcal{F}_{k-1}\big) \geqslant \rho \tag{3.32}$$

for all $1 \leqslant k \leqslant l$; then for every increasing event E that is $\mathcal{F}_l$-measurable we have

$$P(E) \geqslant \overline{P_\rho}(E). \tag{3.33}$$

This result appears as Lemma 3 of Falconer and Grimmett [24], although no proof is given there.

Proof: We proceed by induction on l, noting that (3.33) holds when $l = 1$. Suppose that there exists $\rho \in [0,1]$ such that $P(X_k = 1|\mathcal{F}_{k-1}) \geqslant \rho$ for all $1 \leqslant k \leqslant l+1$; then by the induction hypothesis (3.33) holds for all $\mathcal{F}_l$-measurable increasing events E.

Now let E be an increasing event which is $\mathcal{F}_{l+1}$-measurable. We can write E as a disjoint union

$$E = \big(E_0 \cap \{X_{l+1} = 0\}\big) \cup \big(E_1 \cap \{X_{l+1} = 1\}\big) \tag{3.34}$$

where E_0 and E_1 are both $\mathcal{F}_l$-measurable. Since E is increasing, both E_0 and E_1 are also increasing; moreover $E_0 \subseteq E_1$. Then

$$E = E_0 \cup \big((E_1 \backslash E_0) \cap \{X_{l+1} = 1\}\big) \tag{3.35}$$

and hence

$$\begin{aligned} P(E) &= P(E_0) + P\big((E_1\backslash E_0) \cap \{X_{l+1} = 1\}\big) \\ &\geqslant P(E_0) + \rho P(E_1\backslash E_0) && \text{by (3.32)} \\ &= (1-\rho)P(E_0) + \rho P(E_1) \\ &\geqslant (1-\rho)\overline{P_\rho}(E_0) + \rho\overline{P_\rho}(E_1) && \text{by (3.33)} \\ &= \overline{P_\rho}(E_0) + \rho\overline{P_\rho}(E_1\backslash E_0) \\ &= \overline{P_\rho}(E_0) + \overline{P_\rho}\big((E_1\backslash E_0) \cap \{X_{l+1} = 1\}\big) && \text{since } \overline{P_\rho} \text{ is Bernoulli} \\ &= \overline{P_\rho}(E) \end{aligned} \tag{3.36}$$

completing the proof.

■

Let $\Sigma^{(n)} = \Sigma_0 \cup \ldots \cup \Sigma_n$. We place an ordering on $\Sigma^{(n)}$ as follows. For each $m \leqslant n$, let $\mathbf{I}^{(m)}(1) < \mathbf{I}^{(m)}(2) < \ldots < \mathbf{I}^{(m)}(M^{dm})$ be some fixed ordering on Σ_m. For each $\mathbf{I}^{(m)} \in \Sigma_m$ and $\tilde{\mathbf{I}}^{(\tilde{m})} \in \Sigma_{\tilde{m}}$, where $0 \leqslant m, \tilde{m} \leqslant n$, declare $\mathbf{I}^{(m)} < \tilde{\mathbf{I}}^{(\tilde{m})}$ if and only if either $m > \tilde{m}$, or $m = \tilde{m}$ and $\mathbf{I}^{(m)} < \tilde{\mathbf{I}}^{(\tilde{m})}$ in the ordering on Σ_m. In addition, for $\mathbf{I} \in \Sigma^{(n)}$, define $\mathbf{I}^- = \{\tilde{\mathbf{I}} \in \Sigma^{(n)} : \tilde{\mathbf{I}} < \mathbf{I}\}$.

LEMMA 3.13: Let $\pi = (1-\varepsilon)p$ and suppose that M is sufficiently large so that $M^d \exp(-M\sigma_{1-\pi}/5) < \varepsilon$. Then $P_p(1_A[\mathbf{I}] = 1) \geqslant \pi$ for all $\mathbf{I} \in \Sigma^{(n)}$.

Proof: We proceed by induction on the ordering on $\Sigma^{(n)}$. For every $\mathbf{I} \in \Sigma^{(n)}$, we suppose that $P_p\big(1_A[\tilde{\mathbf{I}}] = 1\big) \geqslant \pi$ holds for all $\tilde{\mathbf{I}} < \mathbf{I}$, and show that we then also have

$$P_p\big(1_A[\mathbf{I}] = 1\big) \geqslant \pi. \tag{3.37}$$

Certainly (3.37) is satisfied for the first M^{dn} terms in the ordering on $\Sigma^{(n)}$, because all level-n cubes are good by definition and hence are available with probability $p > \pi$.

For $m < n$, we place an ordering $\mathbf{I}^{(m)}(1) < \cdots < \mathbf{I}^{(m)}(M^{dm})$ on Σ_m as before. We also place an ordering on the set $\{0, \ldots, 4\}^d$ indexing the 5^d blocks $B[\mathbf{I}^{(m)}(j); \mathbf{k}]$ contained in $\mathbf{I}^{(m)}(j)$. Combining the two gives an ordering on the product space $\Sigma_m \times \{0, \ldots, 4\}^d$ of the indices of all the level-m blocks: We have that $[\mathbf{I}^{(m)}; \mathbf{k}] < [\tilde{\mathbf{I}}^{(m)}; \tilde{\mathbf{k}}]$ if and only if either $\mathbf{I}^{(m)} < \tilde{\mathbf{I}}^{(m)}$, or $\mathbf{I}^{(m)} = \tilde{\mathbf{I}}^{(m)}$ and $\mathbf{k} < \tilde{\mathbf{k}}$. We label the level-$m$ blocks in order as $B(1) < B(2) < \cdots < B(5^d M^{dm})$, and for $1 \leqslant l \leqslant 5^d M^{dm}$ let $\Delta(l)$ denote the event $\{B(l)\,\text{is isolated}\}$. Observe that every $\Delta(l)$ is an increasing event determined by the states of the cubes $\mathbf{I}^{(m+1)} \in \Sigma_{m+1}$.

We first examine the probability that the block $B(1) \subseteq C[\mathbf{I}^{(m)}(1)]$ is isolated. Since $\Delta(1)$ is an increasing event, we can apply Lemma 3.12 to the random variables $\{1_A[\mathbf{I}] : \mathbf{I} \in \mathbf{I}^{(m)}(1)^-\}$ to give

$$\begin{aligned} P_p(\Delta(1)) &\geqslant \overline{P_\pi}(\Delta(1)) \\ &\geqslant 1 - (M/5)^d \exp(-M\sigma_{1-\pi}/5) \qquad \text{by (3.30)} \\ &> 1 - \varepsilon/5^d. \end{aligned} \tag{3.38}$$

Next we consider the lth block $B(l)$, $1 < l \leqslant 5^d M^{dm}$. In general, the event

$\Delta(l)$ is not independent of $\{\Delta(k) : k < l\}$. For each l, let

$$T(l) = \left\{k < l : d\big(B(l), B(k)\big) < \frac{2}{5}M^{-m}\right\}$$

and suppose that $\Delta(k)$ holds for every $k \in T(l)$. We note that $\bigcap_{k\in T(l)} \Delta(k)$ is also an increasing event and hence we may use the FKG inequality (Theorem 1.6) to deduce that

$$\begin{aligned}\overline{P_\pi}\left(\Delta(l) \,\middle|\, \bigcap_{k\in T(l)} \Delta(k)\right) &= \frac{\overline{P_\pi}\big(\Delta(l) \cap \bigcap_{k\in T(l)} \Delta(k)\big)}{\overline{P_\pi}\big(\bigcap_{k\in T(l)} \Delta(k)\big)} \\ &\geqslant \overline{P_\pi}\big(\Delta(l)\big) \\ &> 1 - \varepsilon/5^d \qquad (3.39)\end{aligned}$$

by (3.30).

Now we wish to determine the goodness of each level-m cube $C[\mathbf{I}^{(m)}(j)]$, $1 \leqslant j \leqslant M^{dm}$, in order. To decide whether $C[\mathbf{I}^{(m)}(j)]$ is good or not, we need only examine the blocks $B[\mathbf{I}^{(m)}(j); \mathbf{k}]$ such that

$$\operatorname{dist}\big(B[\mathbf{I}^{(m)}(j); \mathbf{k}], P(j)\big) \geqslant 2M^{-m}/5$$

where

$$P(j) = \bigcup_{l<j} \{C[\mathbf{I}^{(m)}(l)] : C[\mathbf{I}^{(m)}(l)] \text{ is not good}\}$$

as before; let $N(j)$ denote the number of such blocks to be examined. We label these blocks as $B_1 < \cdots < B_{N(j)}$ and let Δ_i denote the event $\{B_i \text{ is isolated}\}$. The cube $C[\mathbf{I}^{(m)}(j)]$ is good if and only if all the blocks $B_1, \ldots, B_{N(j)}$ are isolated, and hence

$$\begin{aligned}\overline{P_\pi}\big(C[\mathbf{I}^{(m)}(j)] \text{ is good}\big) &= \overline{P_\pi}\left(\bigcap_{i=1}^{N(j)} \Delta_i\right) \\ &= \overline{P_\pi}(\Delta_1) \times \overline{P_\pi}(\Delta_2 \,|\, \Delta_1) \times \cdots \times \overline{P_\pi}(\Delta_{N(j)} \,|\, \Delta_1, \ldots, \Delta_{N(j)-1}) \\ &> (1 - \varepsilon/5^d)^{N(j)} > 1 - \varepsilon \qquad (3.40)\end{aligned}$$

since $N(j) \leqslant 5^d$, by applying (3.39) to each of the terms in the product. We note that the event $\{C[\mathbf{I}^{(m)}(j)] \text{ is good}\}$ is increasing and determined by the states of the cubes $\mathbf{I} \in \mathbf{I}^{(m)}(j)^-$, hence we can apply Lemma 3.12 to deduce that

$$\begin{aligned}P_p\big(C[\mathbf{I}^{(m)}(j)] \text{ is good}\big) &\geqslant \overline{P_\pi}\big(C[\mathbf{I}^{(m)}(j)] \text{ is good}\big) \\ &> 1 - \varepsilon \qquad (3.41)\end{aligned}$$

by (3.40). Finally we note that

$$\{1_A[\mathbf{I}^{(m)}(j)] = 1\} \;=\; \{C[\mathbf{I}^{(m)}(j)]\,\text{is good}\} \cap \{\omega[\mathbf{I}^{(m)}(j)] = 1\} \tag{3.42}$$

and therefore

$$P_p\big(1_A[\mathbf{I}^{(m)}(j)] = 1\big) \;>\; (1-\varepsilon)p \;=\; \pi \tag{3.43}$$

as required.

■

Proof of Theorem 3.5:

Let $n \geqslant 1$ and choose $\varepsilon > 0$ such that $\pi \equiv (1-\varepsilon)p > 1 - p_c(\mathbb{M}^d)$. By Lemma 3.11, there exists $\sigma_{1-\pi} > 0$ such that $P_{1-\pi}(0 \leftrightarrow \partial B(N)) \leqslant \exp(-N\sigma_{1-\pi})$ for all $N \geqslant 0$, where $P_{1-\pi}$ is the probability measure for site percolation on $\mathbb{M}^d$ and sites are open with probability $1-\pi$. We take $M = M(\varepsilon)$ sufficiently large so that $M^d \exp(-M\sigma_{1-\pi}/5) < \varepsilon$.

By Lemma 3.13, we have $P_p(1_A[\varnothing] = 1) \geqslant \pi$. By definition, the cube $C[\varnothing]$ is available if it is both good and $\omega[\varnothing] = 1$; hence

$$P_p\big(C[\varnothing]\text{ is good}\big) = P_p\big(1_A[\varnothing] = 1\big)/p \geqslant (\pi/p) = 1 - \varepsilon. \tag{3.44}$$

By Proposition 3.9, sheet percolation occurs in C_n if the cube $C[\varnothing]$ is good. The chosen value of M works uniformly for all $n \geqslant 1$, and hence

$$\begin{aligned} P_p(\text{sheet percolation in } C_\infty) &= \lim_{m\to\infty} P_p(\text{sheet percolation in } C_n) \\ &\geqslant 1 - \varepsilon \end{aligned} \tag{3.45}$$

by (3.44). Letting $\varepsilon \to 0$, we deduce that

$$P_p(\text{sheet percolation in } C_\infty^{[M]}) \to 1 \text{ as } M \to \infty. \tag{3.46}$$

■

Chapter 4

Numerical Results

4.1 Distribution of components of varying sizes

In this section we shall consider the connected components of the fractal percolation process in two dimensions. In particular, we investigate the components larger than a point and give an almost sure scaling law for the number of components with diameter of the order of M^{-N} as $N \to \infty$.

Recall the following properties of the random set C_∞, denoting fractal percolation with subdivision index M and retention probability p, from Section 1.1. For $p \leqslant M^{-2}$, we have $C_\infty = \varnothing$ almost surely by Theorem 1.1. For $M^{-2} < p < p_c$, C_∞ is totally disconnected with probability 1; provided that $C_\infty \neq \varnothing$, the number of isolated points in C_∞ is uncountable, almost surely (see Meester [41]). When $p > p_c$, again provided that $C_\infty \neq \varnothing$, there may exist simultaneously percolating components (with diameter therefore of the order of unity), smaller (non-trivial) connected components and also uncountably many isolated points. Meester proved that the expected number of percolating components is at most $\theta(p)^{-1}$ (where $\theta(p)$ is the probability of percolation); from this he deduced that the number of components larger than a point is countable, almost surely.

We define the *diameter* of a non-empty subset E of $\mathbb{R}^2$ by

$$\operatorname{diam}(E) = \sup\{|x-y| \colon x, y \in E\}.$$

For $N \in \mathbb{N}$, we shall say that a connected component $E \subseteq C_\infty$ is of *scale* N if

$$M^{-(N+1)}/4 \leqslant \operatorname{diam}(E) < M^{-N}/4.$$

Let X_N denote the number of components of C_∞ of scale N, and for a subset A of $[0,1]^2$, let $\#_N(A)$ denote the number of components of scale N contained in the interior of A. Our main result in this section concerns the limiting behaviour of X_N as $N \to \infty$.

THEOREM 4.1: With probability 1, there exists a constant $0 \leqslant \kappa < \infty$ such that $X_N \sim \kappa(M^2p)^N$ as $N \to \infty$.

Theorem 4.1 shows that the number of components of scale N does indeed grow at the rate we might naïvely expect, remaining proportional to the number of retained level-N squares. The result is non-trivial since components of scale $n+m$ (where $n \geqslant 0$, $m \geqslant 1$) are not in general contained within a single level-n square, but instead may intersect several; thus the random variables $\{\#_{n+m}(S) \colon S \text{ is a level-}n\text{ square}\}$ are not independent. Our approach is essentially to decompose C_n into disjoint subsets to restore the independence and count the number of components of scale $n+m$ within each subset.

The sequence $\{X_N\}_{N\geqslant 0}$ is a relatively coarse measure of the distribution of the diameters of the connected components. A more refined method would be to look at the behaviour of $h(\alpha,\varepsilon)$ as $\alpha \to -\infty$ and $\varepsilon \to 0$, where

$$h(\alpha,\varepsilon) = \frac{1}{\varepsilon}\#\{\text{components } E \text{ with } \log \operatorname{diam}(E) \in [\alpha, \alpha+\varepsilon]\}.$$

However, we do not address this refinement here.

In preparation for the proof of Theorem 4.1, we make some further definitions. Let $\mu = M^2p$. For $n \geqslant 0$, we let $S^{(n)}$ be the set of level-n squares contained in C_n; we index these squares as $S^{(n)} = \{S_{n,1}, \dots, S_{n,T_n}\}$ where $T_n = \operatorname{card} S^{(n)}$. Each $S_{n,i}$ may be written as $(a_iM^{-n}, b_iM^{-n}) + [0, M^{-n}]^2$ for some integers $0 \leqslant a_i, b_i \leqslant M^n - 1$. For $1 \leqslant i \leqslant T_n$ and $m \geqslant 1$, define

$$Q^m_{n,i} = (a_iM^{-n}, b_iM^{-n}) + \left[\frac{1}{4}M^{-(n+m)}, M^{-n} - \frac{1}{4}M^{-(n+m)}\right]^2$$

and

$$R^m_{n,i} = (a_iM^{-n}, b_iM^{-n}) +$$

$$\left(\left[-\frac{1}{2}M^{-(n+m)}, M^{-n} + \frac{1}{2}M^{-(n+m)}\right]^2 \setminus \left(\frac{1}{2}M^{-(n+m)}, M^{-n} - \frac{1}{2}M^{-(n+m)}\right)^2\right)$$

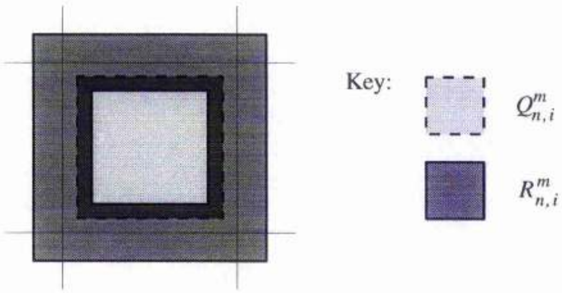


Figure 4.1: Definition of $Q^m_{n,i}$ and $R^m_{n,i}$

as illustrated in Figure 4.1. As m increases, we think of the $Q^m_{n,i}$ approximating $S_{n,i}$ from the inside, whilst the square annuli $R^m_{n,i}$ shrink to approximate $\partial S_{n,i}$. The following result emphasizes this 'sandwich' property.

LEMMA 4.2: For all $n \geqslant 0$, $m \geqslant 1$ and $N \geqslant n+m$, we have

$$\sum_{i=1}^{T_n} \#_N\left(Q^m_{n,i}\right) \leqslant X_N \leqslant \sum_{i=1}^{T_n}\left[\#_N\left(Q^m_{n,i}\right) + \#_N\left(R^m_{n,i}\right)\right].$$

Proof: For the left-hand inequality, simply observe that $\bigcup_{i=1}^{T_n} Q^m_{n,i} \subseteq C_n$ for all $n \geqslant 0$, $m \geqslant 1$. For the right-hand inequality, observe that

$$\bigcup_{i=1}^{T_n}\left[Q^m_{n,i} \cup R^m_{n,i}\right] \supseteq C_n \supseteq C_\infty \tag{4.1}$$

and furthermore, the overlap between $Q^m_{n,i}$ and $R^m_{n,i}$ has width everywhere at least $M^{-(n+m)}/4$. Then every component of C_∞ of scale at least $n+m$ is contained in the interior of at least one of the $Q^m_{n,i}$ or $R^m_{n,i}$.

■

It follows from the work of Meester [41] that $\#_N(Q^m_{n,i})$ and $\#_N(R^m_{n,i})$ both have finite expectation and finite variance. The following lemma establishes an upper bound on the expected value of the $\#_N(R^m_{n,i})$, so that the sandwich inequality of Lemma 4.2 tightens as m increases.

LEMMA 4.3: There exists a constant $0 < v < \infty$ such that for all $n \geqslant 0$, $m \geqslant 1$ and $1 \leqslant i \leqslant T_n$ we have

$$\mathbf{E}\left(\#_{n+m}(R^m_{n,i})\middle|C_n\right) \leqslant 8M^m v.$$

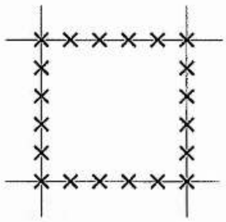


Figure 4.2: Centres of the squares $R \in \mathcal{R}_{n,i}^m$

Proof: Fix $n \geqslant 0$, $m \geqslant 1$ and $1 \leqslant i \leqslant T_n$. Given $S_{n,i} = (a_i M^{-n}, b_i M^{-n}) + [0, M^{-n}]^2$ where $0 \leqslant a_i, b_i \leqslant M^n - 1$, let $\mathcal{R}_{n,i}^m$ denote the set consisting of $8M^m$ squares R of side-length $M^{-(n+m)}$ and with centres equally distributed around the perimeter of $S_{n,i}$ at points

$$\begin{gathered}
\big(a_i M^{-n} + kM^{-(n+m)}/2, b_i M^{-n}\big), \\
\big((a_i+1)M^{-n}, b_i M^{-n} + kM^{-(n+m)}/2\big), \\
\big((a_i+1)M^{-n} - kM^{-(n+m)}/2, (b_i+1)M^{-n}\big), \\
\big(a_i M^{-n}, (b_i+1)M^{-n} - kM^{-(n+m)}/2\big)
\end{gathered}$$

where $k = 0, 1, \ldots, 2M^m - 1$ (see Figure 4.2). Adjacent squares of $\mathcal{R}_{n,i}^m$ overlap by a width of $M^{-(n+m)}/2$ and we can write $R_{n,i}^m = \bigcup_{R \in \mathcal{R}_{n,i}^m} R$, so it is easy to see that

$$\#_{n+m}(R_{n,i}^m) \leqslant \sum_{R \in \mathcal{R}_{n,i}^m} \#_{n+m}(R). \tag{4.2}$$

The random variables $\{\#_{n+m}(R) \colon R \in \mathcal{R}_{n,i}^m\}$ do not have the same distribution, since the level-n neighbours of $S_{n,i}$ may or may not be retained, and the squares R have different positions relative to the level-$(n+m)$ mesh. However, there are at most $2^2 + 2^4 = 20$ different distributions for $\#_{n+m}(R)$, corresponding to whether each of the two or four level-n squares intersecting R is contained in C_n or not. Thus there exists $0 < v < \infty$ such that $\mathbf{E}(\#_{n+m}(R)|C_n) \leqslant v$ for every $R \in \mathcal{R}_{n,i}^m$, for all $n \geqslant 0$, $m \geqslant 1$ and $1 \leqslant i \leqslant T_n$. Summing over $R \in \mathcal{R}_{n,i}^m$, by (4.2) we obtain

$$\mathbf{E}\big(\#_{n+m}(R_{n,i}^m)\big|C_n\big) \leqslant \sum_{R \in \mathcal{R}_{n,i}^m} \mathbf{E}\big(\#_{n+m}(R)\big|C_n\big) \leqslant 8M^m v. \tag{4.3}$$

■

Let $n \geqslant 0$ and $m \geqslant 1$; then since the $\{Q^m_{n,i}\}_{1 \leqslant i \leqslant T_n}$ are mutually disjoint, it is easy to see that the random variables $\{\#_{n+m}(Q^m_{n,i})\}$ are i.i.d. with distribution depending only on m. We define $e_m = \mathbf{E}(\#_{n+m}(Q^m_{n,i})) > 0$ for some representative $Q^m_{n,i}$. However, since the $\{R^m_{n,i}\}_{1 \leqslant i \leqslant T_n}$ may intersect, the random variables $\{\#_{n+m}(R^m_{n,i})\}$ are *not* independent. Instead we define $f_m = 8M^m v$, where v is as in Lemma 4.3, so that $\mathbf{E}(\#_{n+m}(R^m_{n,i})|C_n) \leqslant f_m$ for all $1 \leqslant i \leqslant T_n$.

The next lemma will show that for large enough m, the components of scale $n+m$ do not cluster around the perimeter of $S_{n,i}$ but instead are distributed throughout $S_{n,i}$.

LEMMA 4.4: For $p > 1/M$, we have $f_m/e_m \to 0$ as $m \to \infty$.

Proof: Let $n \geqslant 0$ and $m \geqslant 1$. Given a level-n realisation C_n, the random variables $\{\#_{n+m}(R^m_{n,i})\}_{1 \leqslant i \leqslant T_n}$ do not all have the same distribution, since the level-n squares neighbouring $S_{n,i}$, and therefore intersecting $R^m_{n,i}$, may or may not be contained in C_n. However, there are at most 256 different probability distributions for $\#_{n+m}(R^m_{n,i})$, corresponding to the 2^8 arrangements of the eight neighbours of $S_{n,i}$. For each $m \geqslant 1$, we let Π_m denote the pattern (or a pattern) formed by a square $S_{n,i}$ and its neighbours that maximises $\mathbf{E}(\#_{n+m}(R^m_{n,i}))$ and let v_m denote this maximal value.

Fix $n \geqslant 0$, $1 \leqslant i \leqslant T_n$ and let $m \geqslant 2$. The idea is to show that for large m, copies of Π_m occur many more times in the interior of $S_{n,i}$ than there are level-$(n+m)$ squares around the perimeter of $S_{n,i}$. For the purposes of the following argument, we shall assume that $M \geqslant 3$, although the same result holds for $M = 2$.

Fix a level-$(n+1)$ subsquare Q of $S_{n,i}$ satisfying $Q \subseteq Q^1_{n,i} \cap C_{n+1}$, so that Q is a level-$(n+1)$ retained square in the interior of $S_{n,i}$, observing that such a square exists with positive probability. For $m \geqslant 1$, let W_m denote the number of level-$(n+m)$ subsquares of Q contained in C_{n+m}. (If no square Q exists satisfying $Q \subseteq Q^1_{n,i} \cap C_{n+1}$, we instead define $W_m = 0$ for all $m \geqslant 1$.) Now $\{W_m\}_{m \geqslant 1}$ is branching process with expected family size μ; since $p > 1/M$, the process has a positive probability of survival, *i.e.* $P_p(W_m \not\to 0) > 0$. When the branching process does survive, by Proposition 1.10 there exists almost surely a (random) constant $0 < c < \infty$ such that $W_m \sim c\mu^m$ as $m \to \infty$. We deduce that

$$\eta = \lim_{m\to\infty} \mathbf{E}(W_m/\mu^m)$$

exists and is positive.

Let $m \geqslant 2$; then for every level-$(n+m-1)$ square $S \subseteq Q \cap C_{n+m-1}$, there is a non-zero probability, say π_m, that the level-$(n+m)$ subsquares of S contain a copy of the pattern Π_m. We let W_m^Π denote the total number of such copies of Π_m found in $Q \cap C_{n+m}$. Observe that since there are only finitely many (at most 256) possibilities for Π_m, the π_m are uniformly bounded away from 0, say $\pi_m \geqslant \underline{\pi} = \min\{\pi_m : m \geqslant 2\} > 0$ for all m, and therefore

$$\liminf_{m\to\infty} \frac{\mathbf{E}(W_m^\Pi)}{\mu^m} \geqslant \underline{\pi} \lim_{m\to\infty} \frac{\mathbf{E}(W_m)}{\mu^m} = \eta\underline{\pi} > 0. \tag{4.4}$$

The expected number of components of scale $n+m$ completely within each copy of Π_m is precisely v_m by definition, and so for all $m \geqslant 2$ we have

$$e_m = \mathbf{E}\big(\#_{n+m}(Q_{n,i}^m)\big) \geqslant \mathbf{E}\big(\#_{n+m}(Q)\big) \geqslant v_m \mathbf{E}\big(W_m^\Pi\big) \geqslant \underline{v}\mathbf{E}\big(W_m^\Pi\big) \tag{4.5}$$

where $\underline{v} = \min\{v_m : m \geqslant 2\}$. Therefore

$$\liminf_{m\to\infty} \frac{e_m}{\mu^m} \geqslant \underline{v} \liminf_{m\to\infty} \frac{\mathbf{E}\big(W_m^\Pi\big)}{\mu^m} \geqslant \underline{v}\eta\underline{\pi} > 0 \tag{4.6}$$

by (4.4) and (4.5). Finally observe that since $p > 1/M$, we have $\mu = M^2 p > M$; recalling that $f_m = 8M^m v$, we see that

$$0 \leqslant \limsup_{m\to\infty} \frac{f_m}{e_m} \leqslant \limsup_{m\to\infty} \frac{8M^m v}{\underline{v}\eta\underline{\pi}\mu^m} = 0. \tag{4.7}$$

■

The next two lemmas analyse the limiting behaviour of X_{n+m}/T_n as $n \to \infty$ for $m \geqslant 1$.

LEMMA 4.5: For all $m \geqslant 1$, we have

$$P_p\left(\{C_\infty = \varnothing\} \cup \left\{\liminf_{n\to\infty} \frac{X_{n+m}}{T_n} \geqslant e_m\right\}\right) = 1.$$

Proof: Fix $m \geqslant 1$. For $n \geqslant 0$, define

$$U_n = \begin{cases} \frac{1}{T_n}\sum_{i=1}^{T_n} \#_{n+m}(Q_{n,i}^m) & T_n \geqslant n^2 \\ e_m & T_n < n^2. \end{cases}$$

Let $\varepsilon > 0$ and assume that $T_n > 0$. The random variables $\{\#_{n+m}(Q^m_{n,i})\}_{1\leqslant i\leqslant T_n}$ are i.i.d. with expectation e_m and finite variance, so by Chebyshev's inequality we have

$$P_p\left(\left|\tfrac{1}{T_n}\sum_{i=1}^{T_n}\#_{n+m}(Q^m_{n,i}) - e_m\right| \geqslant \varepsilon \,\Big|\, T_n = t\right) \leqslant \frac{\sigma^2}{t\varepsilon^2} \tag{4.8}$$

for all $t \geqslant 1$, where $\sigma^2 = \mathrm{Var}(\#_{n+m}(Q^m_{n,i}))$. Since

$$|U_n - e_m| = \begin{cases} \left|\frac{1}{T_n}\sum_{i=1}^{T_n}\#_{n+m}(Q^m_{n,i}) - e_m\right| & T_n \geqslant n^2 \\ 0 & T_n < n^2 \end{cases} \tag{4.9}$$

we deduce that

$$\begin{aligned} P_p(|U_n - e_m| \geqslant \varepsilon) &= \sum_{t\geqslant 0} P_p(|U_n - e_m| \geqslant \varepsilon \,|\, T_n = t) P_p(T_n = t) \\ &\leqslant \sum_{t\geqslant n^2} \frac{\sigma^2}{t\varepsilon^2} P_p(T_n = t) \leqslant \frac{\sigma^2}{n^2\varepsilon^2}. \end{aligned} \tag{4.10}$$

By the Borel–Cantelli Lemma, therefore,

$$P_p(|U_n - e_m| \geqslant \varepsilon \text{ i.o.}) = 0 \tag{4.11}$$

(where 'i.o.' means for infinitely many n). Since (4.11) holds for all $\varepsilon > 0$, we deduce that

$$P_p(U_n \to e_m) = 1. \tag{4.12}$$

By Proposition 1.10, if $C_\infty \neq \varnothing$ then $\liminf_{n\to\infty}(T_n/\mu^n) > 0$, almost surely. Intersecting this event with (4.12), we obtain

$$P_p\Big(\{C_\infty = \varnothing\} \cup \big(\{U_n \to e_m\} \cap \{\liminf(T_n/\mu^n) > 0\}\big)\Big) = 1. \tag{4.13}$$

On the event $\{U_n \to e_m\} \cap \{\liminf(T_n/\mu^n) > 0\}$ we have $T_n \geqslant n^2$ for all sufficiently large values of n, and therefore

$$\frac{1}{T_n}\sum_{i=1}^{T_n}\#_{n+m}(Q^m_{n,i}) \to e_m. \tag{4.14}$$

By Lemma 4.2, we have $X_{n+m} \geqslant \sum_{i=1}^{T_n}\#_{n+m}(Q^m_{n,i})$; hence whenever (4.14) holds we have

$$\liminf_{n\to\infty}\frac{X_{n+m}}{T_n} \geqslant e_m. \tag{4.15}$$

Finally by (4.13) we conclude that

$$P_p\left(\{C_\infty = \varnothing\} \cup \left\{\liminf_{n\to\infty}(X_{n+m}/T_n) \geqslant e_m\right\}\right) = 1. \tag{4.16}$$

■

LEMMA 4.6: For all $m \geqslant 1$, we have

$$P_p\left(\{C_\infty = \varnothing\} \cup \left\{\limsup_{n\to\infty} \frac{X_{n+m}}{T_n} \leqslant e_m + f_m\right\}\right) = 1.$$

Proof: Fix $m \geqslant 1$. For $n \geqslant 0$, unlike the $\{\#_{n+m}(Q^m_{n,i})\}_{1\leqslant i\leqslant T_n}$, the random variables $\{\#_{n+m}(R^m_{n,i})\}_{1\leqslant i\leqslant T_n}$ are not i.i.d., so the corresponding version of (4.8) does not hold. However, observe that we may partition the set $B = \{R^m_{n,i}\}_{1\leqslant i\leqslant T_n}$ into at most four subsets $B_1, \dots, B_4$ such that for every $1 \leqslant j \leqslant 4$, the square annuli $R^m_{n,i} \in B_j$ are mutually disjoint. Then we see that the random variables $\{\#_{n+m}(R^m_{n,i}) : R^m_{n,i} \in B_j\}$ are independent, but still possibly with differing distributions.

There are at most $2^8 = 256$ different distributions for the $\#_{n+m}(R^m_{n,i})$, corresponding to whether each of the eight level-n squares neighbouring $S_{n,i}$ is present in C_n or not. We may therefore further partition each B_j, $1 \leqslant j \leqslant 4$, into $B_{j,1}, \dots, B_{j,256}$ such that within each $B_{j,k}$, all the $\#_{n+m}(R^m_{n,i})$ have the same distribution. The random variables $\{\#_{n+m}(R^m_{n,i}) : R^m_{n,i} \in B_{j,k}\}$ are then genuinely i.i.d. for every $1 \leqslant j \leqslant 4$ and $1 \leqslant k \leqslant 256$, and so we are able to apply Chebyshev's inequality to each family $B_{j,k}$ separately.

For $n \geqslant 0$, define

$$U'_n = \begin{cases} \dfrac{1}{T_n}\displaystyle\sum_{i=1}^{T_n} \#_{n+m}(R^m_{n,i}) = \dfrac{1}{T_n}\sum_{j=1}^{4}\sum_{k=1}^{256}\sum_{R^m_{n,i}\in B_{j,k}} \#_{n+m}(R^m_{n,i}) & T_n \geqslant n^2 \\ f_m & T_n < n^2. \end{cases}$$

Let $\varepsilon > 0$ and assume that $T_n > 0$. Analogously to the proof of Lemma 4.5, we obtain

$$P_p(U'_n - f_m \geqslant \varepsilon \text{ i.o.}) = 0 \tag{4.17}$$

since each $\#_{n+m}(R^m_{n,i})$ has expectation at most f_m by Lemma 4.3. Hence $\limsup_{n\to\infty} U'_n \leqslant f_m$ with probability 1 and, on intersecting this event with (4.13), we obtain

$$P_p\Big(\{C_\infty = \varnothing\} \cup \big(\{U_n \to e_m\} \cap \{\limsup U'_n \leqslant f_m\} \cap \{\liminf(T_n/\mu^n) > 0\}\big)\Big) = 1. \tag{4.18}$$

When $\{U_n \to e_m\}$, $\{\limsup U'_n \leqslant f_m\}$ and $\{\liminf(T_n/\mu^n) > 0\}$ all occur simultaneously, we have

$$\lim_{n\to\infty} \frac{1}{T_n}\sum_{i=1}^{T_n} \#_{n+m}(Q^m_{n,i}) = e_m \tag{4.19}$$

and

$$\limsup_{n\to\infty} \frac{1}{T_n} \sum_{i=1}^{T_n} \#_{n+m}(R_{n,i}^m) \leqslant f_m. \tag{4.20}$$

Since $X_{n+m} \leqslant \sum_{i=1}^{T_n} \left[\#_{n+m}(Q_{n,i}^m) + \#_{n+m}(R_{n,i}^m)\right]$ by Lemma 4.2, we deduce from (4.18) that

$$P_p\Big(\{C_\infty = \varnothing\} \cup \Big\{\limsup_{n\to\infty}(X_{n+m}/T_n) \leqslant e_m + f_m\Big\}\Big) = 1. \tag{4.21}$$

∎

Finally we complete the proof of Theorem 4.1 by piecing together the information about the limiting behaviour of X_{n+m}/T_n and f_m/e_m supplied by Lemmas 4.4–4.6.

Proof of Theorem 4.1:

Conditional on $C_\infty \neq \varnothing$, for all $m \geqslant 1$ we have

$$e_m \leqslant \liminf_{n\to\infty} \frac{X_{n+m}}{T_n} \leqslant \limsup_{n\to\infty} \frac{X_{n+m}}{T_n} \leqslant e_m + f_m \tag{4.22}$$

by Lemmas 4.5 and 4.6. Now observe that for all $m \geqslant 1$, again conditional on $C_\infty \neq \varnothing$, we have

$$\begin{aligned} \liminf_{N\to\infty} \frac{X_N}{T_N} &= \liminf_{n\to\infty} \left(\frac{X_{n+m}}{T_n} \frac{T_n}{T_{n+m}}\right) \\ &\geqslant \left(\liminf_{n\to\infty} \frac{X_{n+m}}{T_n}\right) \left(\liminf_{n\to\infty} \frac{T_n}{T_{n+m}}\right) \\ &\geqslant e_m \mu^{-m} > 0 \end{aligned} \tag{4.23}$$

almost surely, by (4.22) and Proposition 1.10. Similarly, for all $m \geqslant 1$ we have

$$\limsup_{N\to\infty} \frac{X_N}{T_N} \leqslant (e_m + f_m)\mu^{-m} < \infty \tag{4.24}$$

almost surely. Recalling that $f_m/e_m \to 0$ as $m \to \infty$ by Lemma 4.4, we see that $\lim_{m\to\infty} e_m \mu^{-m}$ exists and is positive and finite and that

$$\liminf_{N\to\infty} \frac{X_N}{T_N} = \limsup_{N\to\infty} \frac{X_N}{T_N} = \lim_{m\to\infty} e_m \mu^{-m} \tag{4.25}$$

with probability 1, conditional on $C_\infty \neq \varnothing$.

To conclude, therefore, either $C_\infty = \varnothing$, in which case $\lim_{N\to\infty} X_N = 0$, or by Proposition 1.10 there exists (with probability 1) a random number $c > 0$ such that $\lim_{N\to\infty}(T_N/\mu^N) = c$ and hence

$$\lim_{N\to\infty} \frac{X_N}{\mu^N} = \lim_{N\to\infty} \left(\frac{X_N}{T_N} \frac{T_N}{\mu^N}\right) = c \lim_{m\to\infty} e_m \mu^{-m} \tag{4.26}$$

almost surely. Setting $\kappa = c\lim_{m\to\infty} e_m\mu^{-m}$ completes the proof.

■

4.2 A rigorous upper bound on the percolation function

Consider the usual fractal percolation process in $[0,1]^2$ with subdivision index $M = 3$ and retention probability $0 \leqslant p \leqslant 1$, denoting the pre-fractal sets by C_n and the limiting set by C_∞. In this section we describe calculations, performed with the aid of a computer algorithm and Maple computer arithmetic, that give a rigorous and useful upper bound on the percolation function $\theta(p) = P_p(\text{percolation in } C_\infty)$.

We say that a *level-n crack* exists if the closure of the complement of C_n contains a continuous path intersecting both top and bottom sides of the unit square, *i.e.* there exists a continuous function $\gamma\colon[0,1] \to \overline{[0,1]^2\backslash C_n}$ such that $\gamma(0) \in [0,1]\times\{0\}$ and $\gamma(1) \in [0,1]\times\{1\}$. Let $f_n(p) = P_p(\exists \text{ level-}n \text{ crack})$. If there is no level-$n$ crack, then certainly percolation occurs in C_n. The converse is not the case: The existence of a level-n crack does not by itself preclude the occurrence of percolation in C_n, since it is possible for percolating paths to pass between squares that only intersect at a single point, as illustrated in Figure 4.3. However, as shown by Falconer and Grimmett [24], such a percolating path in

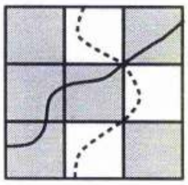


Figure 4.3: A crack and a percolating path existing simultaneously

C_n is broken at higher levels with probability 1, and so from (1.7) we deduce that

$$\lim_{n\to\infty} f_n(p) = \lim_{n\to\infty} P_p(\text{no percolation in } C_n) = 1 - \theta(p). \qquad (4.27)$$

We shall obtain a lower bound on $\lim_{n\to\infty} f_n(p)$, thus giving an upper bound on $\theta(p)$.

First we calculate $f_1(p)$, the probability of a level-1 crack. If fewer than three of the nine level-1 squares are removed, then no crack can exist; if more than six are removed, a crack will certainly exist. If the number of squares removed is between three and six, a crack may or may not exist, depending upon the arrangement of the squares removed. Since there are 2^9 possible configurations, a computer was required to examine each configuration in turn and count those in which a crack exists. We find that

$$\begin{aligned} f_1(p) &= (1-p)^9 + 9p(1-p)^8 + 36p^2(1-p)^7 + \\ &\quad 81p^3(1-p)^6 + 104p^4(1-p)^5 + 67p^5(1-p)^4 + 17p^6(1-p)^3 + \\ &\quad 0p^7(1-p)^2 + 0p^8(1-p) + 0p^9. \end{aligned} \tag{4.28}$$

Next, for $n \geqslant 1$, we wish to estimate $f_{n+1}(p)$ in terms of $f_n(p)$; to do this, we introduce a modified model as follows:

S_1	S_2	S_3
S_4	S_5	S_6
S_7	S_8	S_9

Figure 4.4: Labelling the level-1 squares

Label the nine level-1 squares as $S_1, \ldots, S_9$ in the manner shown in Figure 4.4. Let $\alpha \in [0,1]$. To each square S_i, $1 \leqslant i \leqslant 9$, we assign a *type* independently at random with probabilities as follows:

- Type O (open) — probability $1-p$
- Type V (vertical) — probability $p\alpha(1-\alpha)$
- Type H (horizontal) — probability $p(1-\alpha)\alpha$
- Type B (both) — probability $p\alpha^2$
- Type C (closed) — probability $p(1-\alpha)^2$.

We think of the state of square S_i as corresponding to the ability of S_i to transmit a crack: Type O squares are vacant and thus will transmit all cracks, whilst the level-1 retained squares are of types V, H, B or C. The arrangement of subsquares of a type V square S_i is such that there exists a crack between the top and bottom sides of S_i (but not from left to right); similarly if S_i is of type H then there is a crack between left and right sides of S_i (but not from top to bottom). Finally, S_i is of type B if there are both horizontal and vertical cracks, and of type C if there are neither.

We define the state space of all possible arrangements of the five types for $S_1, \dots, S_9$ by $\Xi = \{\mathrm{O,V,H,B,C}\}^9$. The probability measure $P_{p,\alpha}$ on subsets of Ξ is the natural product probability measure.

Given a configuration $\xi \in \Xi$, we use the following rules for deciding whether a crack is transmitted from $[0,1] \times \{1\}$ to $[0,1] \times \{0\}$. The areas above squares S_1, S_2, S_3 and below squares S_7, S_8, S_9 are regarded as open (type O). A crack is transmitted:

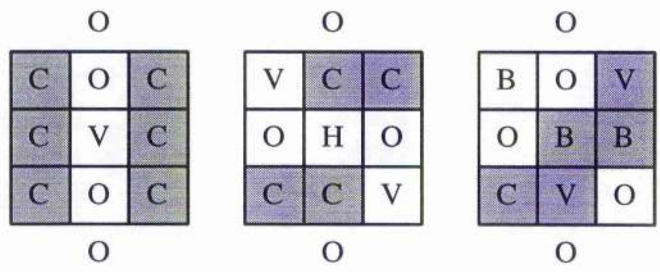
- from a type O square to any vertically, horizontally or diagonally adjacent type O square
- from a type O square vertically through a type V or B square to another type O square
- from a type O square horizontally through a type H or B square to another type O square

No other combinations are allowed; in particular, no crack may be transmitted through a type C square, or two vertically adjacent type V squares, or two horizontally adjacent type H squares. By way of illustration, under these rules a crack is transmitted from $[0,1] \times \{1\}$ to $[0,1] \times \{0\}$ in Figures 4.5a and 4.5b but not in Figure 4.5c. Observe that setting up the model in this way introduces a degree of inefficiency; for example, in the configuration shown in Figure 4.5c, it is possible that a crack between $[0,1] \times \{1\}$ and $[0,1] \times \{0\}$ could exist if the vacant subsquares of S_5 and S_8 matched up, but this is not guaranteed.

Now define the function $g\colon [0,1] \times [0,1] \to [0,1]$ by

$$g(p,\alpha) = \sum_{\xi \in \Xi} P_{p,\alpha}(\xi) I(\xi)$$

where $I(\xi)$ takes the value 1 if a crack is transmitted from $[0,1]\times\{1\}$ to $[0,1]\times\{0\}$ according to the rules above, and $I(\xi) = 0$ otherwise. A computer algorithm written in Pascal, Program `find_g` in Appendix A.1, was used to examine each



(a) (b) (c)

Figure 4.5: Cracks are transmitted in (a) and (b) but not in (c)

of the 5^9 configurations $\xi \in \Xi$ in turn and count those in which a crack is transmitted. (In fact, the number of configurations needing to be checked can be reduced considerably to $3^8.5$, since we need to consider horizontal cracks only in the case of the central square S_5.)

The output of Program `find_g` is a table of coefficients of the powers of p, $1-p$, α and $1-\alpha$ in the polynomial $g(p,\alpha)$; see Table 4.1.

Using Maple computer arithmetic, this polynomial simplifies considerably and we obtain

$$\begin{aligned} g(p,\alpha) &= (1-p)^9 + 9p(1-p)^8 + 36p^2(1-p)^7 + \\ &\quad p^3(1-p)^6(81+7\alpha-6\alpha^2+2\alpha^3) + \\ &\quad p^4(1-p)^5(104+38\alpha-24\alpha^2+6\alpha^3) + \\ &\quad p^5(1-p)^4(67+84\alpha-42\alpha^2+8\alpha^3) + \\ &\quad p^6(1-p)^3(17+60\alpha+9\alpha^2-32\alpha^3+19\alpha^4-6\alpha^5+\alpha^6) + \\ &\quad p^7(1-p)^2(15\alpha+12\alpha^2-2\alpha^3-5\alpha^4+2\alpha^5)+3p^8(1-p)\alpha^2. \quad (4.29) \end{aligned}$$

A three-dimensional plot of $g(p,\alpha)$ is shown in Figure 4.6. Note that g is a decreasing function of p and an increasing function of α, since increasing either $(1-p)$ or α increases the probability of transmission of a crack. Observe also that taking $\alpha = 0$, we have $g(p,0) = f_1(p)$ by (4.28). For consistency we shall define $f_0(p) = 0$ for all $0 \leqslant p \leqslant 1$.

$g(p, \alpha) =$

```
p^0 *(1-p)^9 * ( 1*a^0 *(1-a)^0) +
p^1 *(1-p)^8 * ( 8*a^0 *(1-a)^1+  1*a^0 *(1-a)^2+  8*a^1 *(1-a)^0+
2*a^1 *(1-a)^1+  1*a^2 *(1-a)^0) +
p^2 *(1-p)^7 * (28*a^0 *(1-a)^2+  8*a^0 *(1-a)^3+ 56*a^1 *(1-a)^1+
24*a^1 *(1-a)^2+ 28*a^2 *(1-a)^0+ 24*a^2 *(1-a)^1+  8*a^3 *(1-a)^0) +
p^3 *(1-p)^6 * (54*a^0 *(1-a)^3+ 27*a^0 *(1-a)^4+ 168*a^1 *(1-a)^2+
109*a^1 *(1-a)^3+ 168*a^2 *(1-a)^1+ 165*a^2 *(1-a)^2+ 56*a^3 *(1-a)^0+
111*a^3 *(1-a)^1+ 28*a^4 *(1-a)^0) +
p^4 *(1-p)^5 * (60*a^0 *(1-a)^4+ 44*a^0 *(1-a)^5+ 266*a^1 *(1-a)^3+
232*a^1 *(1-a)^4+ 416*a^2 *(1-a)^2+ 486*a^2 *(1-a)^3+ 280*a^3 *(1-a)^1+
506*a^3 *(1-a)^2+ 70*a^4 *(1-a)^0+ 262*a^4 *(1-a)^1+ 54*a^5 *(1-a)^0) +
p^5 *(1-p)^4 * (36*a^0 *(1-a)^5+ 31*a^0 *(1-a)^6+ 224*a^1 *(1-a)^4+
226*a^1 *(1-a)^5+ 506*a^2 *(1-a)^3+ 653*a^2 *(1-a)^4+ 540*a^3 *(1-a)^2+
974*a^3 *(1-a)^3+ 278*a^4 *(1-a)^1+ 799*a^4 *(1-a)^2+ 56*a^5 *(1-a)^0+
344*a^5 *(1-a)^1+ 61*a^6 *(1-a)^0) +
p^6 *(1-p)^3 * ( 9*a^0 *(1-a)^6+  8*a^0 *(1-a)^7+ 82*a^1 *(1-a)^5+
88*a^1 *(1-a)^6+ 276*a^2 *(1-a)^4+ 368*a^2 *(1-a)^5+ 444*a^3 *(1-a)^3+
788*a^3 *(1-a)^4+ 376*a^4 *(1-a)^2+ 956*a^4 *(1-a)^3+ 162*a^5 *(1-a)^1+
668*a^5 *(1-a)^2+ 28*a^6 *(1-a)^0+ 252*a^6 *(1-a)^1+ 40*a^7 *(1-a)^0) +
p^7 *(1-p)^2 * ( 6*a^1 *(1-a)^6+  9*a^1 *(1-a)^7+ 42*a^2 *(1-a)^5+
69*a^2 *(1-a)^6+ 116*a^3 *(1-a)^4+ 227*a^3 *(1-a)^5+ 164*a^4 *(1-a)^3+
410*a^4 *(1-a)^4+ 126*a^5 *(1-a)^2+ 437*a^5 *(1-a)^3+ 50*a^6 *(1-a)^1+
275*a^6 *(1-a)^2+  8*a^7 *(1-a)^0+ 95*a^7 *(1-a)^1+ 14*a^8 *(1-a)^0) +
p^8 *(1-p)^1 * ( 1*a^2 *(1-a)^6+  2*a^2 *(1-a)^7+  6*a^3 *(1-a)^5+
14*a^3 *(1-a)^6+ 15*a^4 *(1-a)^4+ 42*a^4 *(1-a)^5+ 20*a^5 *(1-a)^3+
70*a^5 *(1-a)^4+ 15*a^6 *(1-a)^2+ 70*a^6 *(1-a)^3+  6*a^7 *(1-a)^1+
42*a^7 *(1-a)^2+  1*a^8 *(1-a)^0+ 14*a^8 *(1-a)^1+  2*a^9 *(1-a)^0) +
p^9 *(1-p)^0 * ( 0)
```

Table 4.1: Output from Program find_g

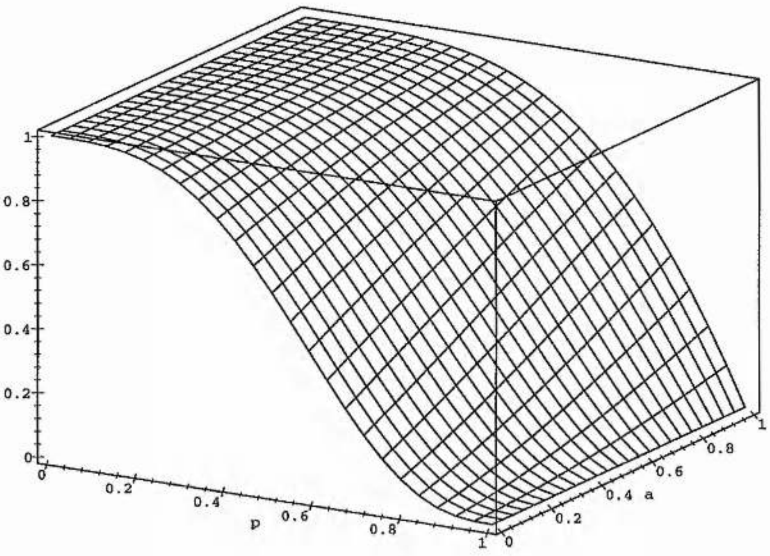


Figure 4.6: The polynomial $g(p, \alpha)$

LEMMA 4.7: For all $n \geqslant 0$, we have

$$f_{n+1}(p) \geqslant g\big(p, f_n(p)\big). \tag{4.30}$$

Proof: We use the modified model described above, setting $\alpha = f_n(p)$, to estimate $f_{n+1}(p)$. By rescaling, the probability that a vertical level-$(n+1)$ crack exists in a retained level-1 square is exactly α; thus each level-1 square S_i, $1 \leqslant i \leqslant 9$, is of type O with probability $1-p$, type V or B with probability $p\alpha$ and type H or B with probability $p\alpha$. From the FKG inequality (Theorem 1.6) inequality applied to the decreasing events $\{\exists$ vertical crack$\}$, $\{\exists$ horizontal crack$\}$, we deduce that S_i is of type B with probability at least $p\alpha^2$ and of type C with probability at most $p(1-\alpha)^2$. The probability that a level-$(n+1)$ crack exists is therefore at least the probability that a crack is transmitted according to the rules in the modified model with $\alpha = f_n(p)$, *i.e.* $f_{n+1}(p) \geqslant g(p, f_n(p))$.

■

For $0 \leqslant p \leqslant 1$, define an iterative process by

$$g_0(p) = 0, \quad g_{n+1}(p) = g\big(p, g_n(p)\big).$$

Since $g(p,\alpha)$ is increasing in α, the limit $g_*(p) = \lim_{n\to\infty} g_n(p)$ exists and is the (smallest) solution to $g_*(p) = g(p, g_*(p))$. Applying Lemma 4.7 repeatedly, we deduce that $f_n(p) \geqslant g_n(p)$ for all n, and hence

$$\lim_{n\to\infty} f_n(p) \geqslant g_*(p). \tag{4.31}$$

Finally by (4.27) we have

$$\theta(p) \leqslant 1 - g_*(p), \tag{4.32}$$

and this is the upper bound illustrated in Figure 4.7 (along with the curves $1 - g_n(p)$ for $1 \leqslant n \leqslant 3$).

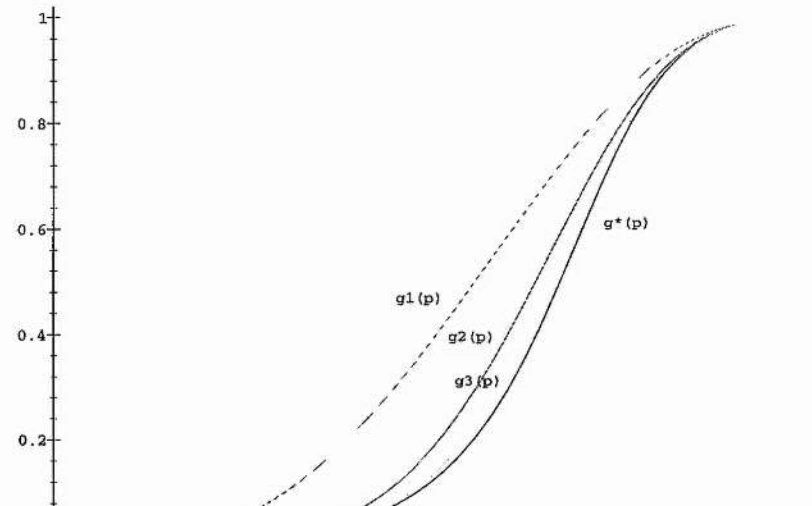


Figure 4.7: An upper bound on $\theta(p)$

The techniques in this section may theoretically be used in conjunction with more careful estimates of the probabilities of cracks to give an improved upper bound on $\theta(p)$ — for example, we could let the polynomial $f_1(p)$ in (4.27) represent the exact probability that a level-2 crack exists. However, such improvements would quickly become computationally very intensive; in this example; the algorithm would have to search through 5^{81} configurations and the resulting polynomial $g(p,\alpha)$ could have up to 81^2 terms.

4.3 Computer simulations

Finally for this chapter we look at the results of some computer simulations of the fractal percolation process. The motivation for these simulations was an attempt to investigate how the number and sizes of the connected components varies with p, and ultimately produce improved estimates for the critical probability p_c. The bounds on p_c obtained by Chayes *et al.* [8] are far from tight; from Theorem 1.3, we have $p_c \leqslant 1 - M^{-5}/3$, whilst a simple branching process argument shows that $p_c \geqslant 1/\sqrt{M}$. Some improvements have been made by refining the techniques of Chayes *et al.*; the best known bounds on $p_c(3)$ are currently $0.6346 < p_c(3) < 0.9830$, from Wu and Liu [58] and Xu and Su [59] respectively. Note that all of the arguments in this section should be regarded as heuristic and based on experimental evidence only.

In the case of discrete (bond or site) percolation in $d \geqslant 2$ dimensions, good estimates for the critical probability may be made from an examination of the probabilities of crossings of large boxes. For $N \geqslant 1$, let $B(N)$ be the box of side-length $2N$ centred at the origin and let $LR(N)$ denote the event {there exists a left-right open crossing of $B(N)$}. Recall from Lemma 3.11 that when p is sub-critical, the probability that the origin is connected by an open path to the surface of $B(N)$ decays exponentially as $N \to \infty$; hence we also have $P_p(LR(N)) \to 0$. In contrast, when p is super-critical, we have $P_p(LR(N)) \to 1$ as $N \to \infty$ (see Theorem 6.125 of Grimmett [29]), and it is believed that the convergence here is also exponential. Thus a sharp dichotomy can be observed at the critical point in the limiting behaviour of $P_p(LR(N))$.

The situation is more complicated in the case of fractal percolation. If we simulate the process as far as a finite level n and find that a percolating component exists, that component may well be broken with probability 1 at some higher level. Conversely, even if we perform the simulation many times and find no percolation occurring, there is no guarantee that $\theta(p) = 0$ and $p < p_c$. We thus have to look closer into the nature of the connected components at finite levels n in our attempt to determine whether a realisation of the process is sub- or super-critical.

It turns out that two quantities are of particular use for this purpose. For $0 \leqslant p \leqslant 1$, $n \geqslant 1$ and a realisation of C_n, let $\chi_n(p)$ denote the maximum number of level-n squares contained in a connected component of C_n and let $\delta_n(p)$ be the maximum diameter of a connected component of C_n. (We measure diameter in the L_1 metric and do not consider two squares meeting only at a single point to be connected, since we know that C_∞ contains no level-n vertices, almost surely.) Values of $\chi_n(p)$ (rescaled by a factor of M^n) and $\delta_n(p)$ were calculated for a number of realisations of fractal percolation and a range of values of p; some of the results for $M = 3$ are shown in Table 4.2.

Due to limitations of computing time and power, fractal percolation cannot be simulated very far — in the case of $M = 3$ and without sophisticated variable handling, a maximum of six or seven levels can be generated and examined in a reasonable time-frame. The computations were performed with a Pascal program, `clusters`, a schematic of which appears in Appendix A.2.

In order to interpret our results, consider the following situations. First suppose that the process is super-critical, *i.e.* $p \geqslant p_c$. Provided that $C_\infty \neq \varnothing$, we know that the expected number of connected components larger than a point is infinite and countable, almost surely. By Corollary 2.4, each such component E has box-counting dimension greater than 1, so the number of level-n squares contained in E grows faster than M^n as $n \to \infty$, *i.e.* $\lim_{n\to\infty}(\chi_n(p)/M^n) = \infty$. On the other hand, if the process is sub-critical, *i.e.* $p < p_c$, then C_∞ is totally disconnected, almost surely, and we might expect to see $\delta_n(p)$ decreasing exponentially fast to 0 as $n \to \infty$.

We can use these observations to conjecture some tentative bounds for p_c when $M = 3$, based on the experimental values of $\chi_n(p)$ and $\delta_n(p)$ for $1 \leqslant n \leqslant 7$. When $p \leqslant 0.77$, in all the simulations $\delta_n(p)$ decreases at a roughly geometric rate, whilst $\chi_n(p)/3^n$ also shows a tendency to decrease (after perhaps increasing at first for small n). Looking at the distribution of the components, we see that a component at level n typically breaks into several pieces at level $n + 1$; the limit set will therefore be dust-like, and the whole process sub-critical.

When $p \geqslant 0.82$, in every case $\chi_n(p)/3^n$ is increasing in n, whilst the decline (if any) in $\delta_n(p)$ is more slight; typically, there is one large cluster which survives relatively intact as n increases, although some smaller components may break off from the edges. We therefore conjecture that for these values of p, the process is super-critical.

For $0.77 < p < 0.82$, it is more difficult to confidently predict the limit-

p	$n \rightarrow$	1	2	3	4	5	6	7
0.60	$\chi_n(p)/3^n$	0.3333	0.7778	1.3704	0.5062	0.3169	0.2071	0.0933
	$\delta_n(p)$	0.3333	0.3333	0.3333	0.1605	0.0741	0.0366	0.0165
0.65	$\chi_n(p)/3^n$	1.3333	1.2222	1.2963	1.1235	0.5432	0.3882	0.1372
	$\delta_n(p)$	1.0000	0.6667	0.3333	0.2593	0.1070	0.0700	0.0233
0.70	$\chi_n(p)/3^n$	3.0000	3.1111	2.8889	3.9630	1.6132	1.3841	0.4582
	$\delta_n(p)$	1.0000	1.0000	0.6296	0.3704	0.1852	0.0933	0.0412
0.75	$\chi_n(p)/3^n$	2.3333	5.7778	9.4074	12.012	8.6626	6.3800	2.7933
	$\delta_n(p)$	1.0000	1.0000	1.0000	0.7037	0.3704	0.2469	0.0905
0.76	$\chi_n(p)/3^n$	2.3333	2.5556	3.4444	2.8519	5.5556	3.5871	3.1064
	$\delta_n(p)$	1.0000	0.8889	0.6667	0.3704	0.3210	0.2346	0.1033
0.77	$\chi_n(p)/3^n$	2.6667	5.8889	12.556	22.309	41.243	14.601	9.2314
(run 1)	$\delta_n(p)$	1.0000	1.0000	1.0000	1.0000	1.0000	0.3951	0.1811
0.77	$\chi_n(p)/3^n$	2.6667	6.0000	13.407	8.9259	12.392	11.258	8.3827
(run 2)	$\delta_n(p)$	1.0000	1.0000	1.0000	0.8025	0.4815	0.3416	0.1674
0.78	$\chi_n(p)/3^n$	3.0000	6.7778	9.0000	16.383	15.169	20.362	9.6735
(run 1)	$\delta_n(p)$	1.0000	1.0000	1.0000	1.0000	0.6296	0.5062	0.2675
0.78	$\chi_n(p)/3^n$	1.6667	4.3333	10.037	22.840	21.683	18.348	13.340
(run 2)	$\delta_n(p)$	1.0000	1.0000	1.0000	1.0000	0.6667	0.4444	0.2606
0.79	$\chi_n(p)/3^n$	3.0000	3.6667	8.2593	18.012	36.868	36.406	17.465
	$\delta_n(p)$	1.0000	1.0000	1.0000	1.0000	1.0000	0.5583	0.2469
0.80	$\chi_n(p)/3^n$	2.0000	4.1111	3.0000	6.2099	14.444	31.909	48.342
	$\delta_n(p)$	1.0000	1.0000	0.6667	0.6667	0.6296	0.4444	0.4074
0.81	$\chi_n(p)/3^n$	2.3333	4.8889	11.407	18.951	44.490	84.731	53.696
(run 1)	$\delta_n(p)$	1.0000	1.0000	1.0000	1.0000	1.0000	0.9506	0.4074
0.81	$\chi_n(p)/3^n$	3.0000	7.4444	18.519	43.802	31.737	62.716	82.881
(run 2)	$\delta_n(p)$	1.0000	1.0000	1.0000	1.0000	0.7531	0.7284	0.5217
0.82	$\chi_n(p)/3^n$	2.6667	6.8889	16.889	40.840	95.617	143.40	182.18
(run 1)	$\delta_n(p)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.7778
0.82	$\chi_n(p)/3^n$	2.6667	6.8889	17.148	41.667	95.337	193.63	314.65
(run 2)	$\delta_n(p)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.83	$\chi_n(p)/3^n$	3.0000	7.5556	18.963	46.296	107.12	262.25	595.26
	$\delta_n(p)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.84	$\chi_n(p)/3^n$	2.3333	6.1111	15.593	39.420	95.309	237.12	583.06
	$\delta_n(p)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.85	$\chi_n(p)/3^n$	2.6667	7.5556	19.296	48.938	123.74	311.28	783.56
	$\delta_n(p)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.90	$\chi_n(p)/3^n$	2.0000	5.5556	14.741	39.877	105.15	288.60	778.88
	$\delta_n(p)$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 4.2: Results of simulations for $M = 3$

ing behaviour of $\chi_n(p)/3^n$ and $\delta_n(p)$ from the simulations; we see a transition between the two regimes as p varies. We estimate therefore that the critical probability p_c lies in the interval $(0.77, 0.82)$.

Repeating the experiments for $M = 2$ (with n up to 11) and $M = 4$ (with n up to 5), we estimate that $p_c(2) \in (0.82, 0.87)$ and $p_c(4) \in (0.73, 0.77)$, thus lending some weight to the conjecture that p_c is monotonic decreasing in M.

The relatively wide confidence intervals for p_c highlight the inherent fractal nature of the process; it is very difficult to draw any firm conclusions about the limiting set from an analysis, mathematical or numerical, of the first few pre-fractal sets. Perhaps a more efficient use of computing power would be to concentrate on the local properties of the limit set, carrying the construction arbitrarily far, rather than generate the entire set which requires exponentially increasing amounts of time and memory.

Chapter 5

Related Models and Open Questions

5.1 Tiling the plane with random Cantor sets

In this section we consider the following model, proposed by Chayes, Chayes and Durrett [8]. A full definition is given below; briefly, we fix an index parameter $M \geqslant 2$, a retention probability $0 \leqslant p \leqslant 1$ and let C_∞ denote the random Cantor set given by fractal percolation in the unit square $[0,1]^2$. For each point $\mathbf{z} \in \mathbb{Z}^2$, we let $C_\infty[\mathbf{z}]$ be an independent copy of C_∞ translated by $\mathbf{z}$ to lie in the square $\mathbf{z} + [0,1]^2$. We denote the union of all such sets by

$$C'_\infty = \bigcup_{\mathbf{z} \in \mathbb{Z}^2} C_\infty[\mathbf{z}]$$

and the induced probability measure by P_p. The random set C'_∞ thus represents a 'tiling' of the plane by copies of the fractal percolation process.

For a more precise formulation, recall the following definitions from Section 1.1. Let $J = \{0, 1, \ldots, M-1\}^2$, $\Sigma_n = J^n$ and $\Sigma = \bigcup_{n \geqslant 1} \Sigma_n$, so that Σ corresponds to the set of all permitted subsquares of $[0,1]^2$. We further define $\Sigma' = \mathbb{Z}^2 \times \Sigma$ and $\Omega = \{0,1\}^{\Sigma'}$; then Ω' is the state space for the tiling of the plane. For each $\mathbf{I} = (\mathbf{z}, \mathbf{i}_1, \ldots, \mathbf{i}_n) \in \Sigma_n$, $n \geqslant 1$, we define the indicator function $1_\omega[\mathbf{I}]$ by

$$1_\omega[\mathbf{I}] = \prod_{j=1}^{n} \omega\big[(\mathbf{z}, \mathbf{i}_1, \ldots, \mathbf{i}_j)\big].$$

Then elements $\omega \in \Omega'$ represent particular realisations of fractal percolation in the plane, with each $\mathbf{I} \in \Sigma'$ such that $1_\omega[\mathbf{I}] = 1$ corresponding to a retained square, and each $\mathbf{I}$ such that $1_\omega[\mathbf{I}] = 0$ to a vacant square. As in the standard model, the cylinder sets are defined by (1.3) and we let $\mathcal{F}$ denote the σ-algebra generated by the cylinder subsets of Ω'. The natural product probability measure on Ω' is denoted by P_p and is defined by (1.4) in exactly the same way as before.

Theorem 3 of Chayes *et al.* claims that if $p \geqslant p_c$, then C'_∞ has a unique unbounded connected component, almost surely. The proof of the existence of an unbounded component follows from a standard construction of Smythe and Wierman [53]; after first establishing that for $p \geqslant p_c$, the probability of a left to right crossing of $C'_\infty \cap [0, M^n]$ tends to 1 as $n \to \infty$, crossings of ever-increasing rectangles are pieced together to build up an unbounded component. Note that it also follows from their construction that the unbounded component intersects a positive fraction of the unit grid squares.

However, the proof of the uniqueness of the unbounded component is not given there; the authors say that it is proved "in the same way as in the ordinary case" and refer the reader to the early work of Harris [30]. This original proof is very specific to the problem of bond percolation on the square lattice, and relies heavily on the geometry of this graph and its dual: If two or more infinite clusters co-exist, then the dual graph must also contain an infinite cluster; with the knowledge that the critical probability for both the primary and dual processes is at least $1/2$, this event is shown to have probability zero.

In the fractal case, there is no such obvious dual process; instead we shall use a different combination of established techniques. We shall employ ideas from Newman and Schulman [44, 45] and Burton and Keane [5] to complete the uniqueness proof, and also investigate the connectivity of the unbounded connected component.

THEOREM 5.1: For $p \geqslant p_c$, the set C'_∞ contains exactly one unbounded connected component, P_p-almost surely.

In preparation for the proof of Theorem 5.1, we make a few further definitions. Recalling that C_n denotes the nth level of the fractal percolation process in $[0,1]^2$, for each $\mathbf{z} \in \mathbb{Z}^2$ let $C_n[\mathbf{z}]$ be an independent copy of C_n translated to lie in $\mathbf{z} + [0,1]^2$ and let $C'_n = \bigcup_{\mathbf{z} \in \mathbb{Z}^2} C_n[\mathbf{z}]$. For $K \geqslant 0$, we let $B(K) = [-K, K]^2$

and let $A(K)$ denote the square annulus $B(K+1)\backslash\text{int } B(K)$. We shall call a connected component of C'_∞ a *cluster*, and a connected component with infinite diameter an *unbounded cluster*.

5.1.1 Uniqueness of the unbounded cluster

The strategy is as follows: We first show that the probability measure P_p is ergodic with respect to translation by points of $\mathbb{Z}^2$. The events

$$A_q = \{\exists \text{ exactly } q \text{ unbounded clusters}\}$$

are shown to be both measurable and translation invariant, and hence by ergodicity occur with probability 0 or 1. We deduce that the number of unbounded clusters is an almost sure constant. Finally we show that with probability 1, C'_∞ contains clusters encircling the origin with arbitrarily large diameter, and hence in fact there exists at most one unbounded cluster.

The probability measure P is said to be *ergodic* with respect to an invertible transformation T if every set $A \in \mathcal{F}$ for which $T^{-1}(A) = A$ satisfies either $P(A) = 0$ or $P(A) = 1$. The transformation T is said to be *mixing* with respect to P if

$$\lim_{k\to\infty} P(A \cap T^{-k}B) = P(A)P(B) \tag{5.1}$$

for all $A, B \in \mathcal{F}$; as we shall see in the proof of Theorem 5.4, it is well known that if a transformation T and a measure P are mixing, then they are in fact ergodic.

PROPOSITION 5.2: Suppose that (5.1) holds for all cylinder sets A, B. Then T is mixing with respect to P.

Proof: See Billingsley [4], Theorem 1.2.

■

Let $T_\mathbf{z}$, $\mathbf{z} \in \mathbb{Z}^2$, be the transformation acting on elements or subsets of Ω given by translation by $\mathbf{z}$. The next two results show that each $T_\mathbf{z}$ is measure-preserving and mixing, and hence that P_p is ergodic with respect to $T_\mathbf{z}$. Note that Theorem 5.4 is only a slight modification of the analogous standard result for a single transformation.

LEMMA 5.3: Every transformation $T_{\mathbf{z}}$, $\mathbf{z} \in \mathbb{Z}^2$, is measure-preserving for P_p.

Proof: Let A be a cylinder subset of Ω; then $T_{\mathbf{z}}^{-1}A$ is also a cylinder set and clearly $P_p(T_{\mathbf{z}}^{-1}A) = P_p(A)$. By the uniqueness of the extension of measures, this equality holds for all $A \in \mathcal{F}$.

■

THEOREM 5.4: For every $\mathbf{z} \in \mathbb{Z}^2 \backslash \{(0,0)\}$, the measure P_p is ergodic with respect to $T_{\mathbf{z}}$.

Proof: Let A and B be cylinder subsets of Ω and let $\mathbf{z} \in \mathbb{Z}^2 \backslash \{(0,0)\}$. Then for all $k \geqslant 0$, the sets A and $T_{\mathbf{z}}^{-k}B$ are both cylinder sets, specified by the values of $\omega(x)$ for x belonging to finite subsets X, X' of Σ'; for sufficiently large k, the sets X, X' will be disjoint. For these values of k, the events A and $T_{\mathbf{z}}^{-k}B$ are independent, and hence

$$P_p(A \cap T_{\mathbf{z}}^{-k}B) = P_p(A)P_p(T_{\mathbf{z}}^{-k}B) = P_p(A)P_p(B) \tag{5.2}$$

since $T_{\mathbf{z}}$ is measure-preserving by Lemma 5.3. By Proposition 5.2, we deduce that $T_{\mathbf{z}}$ is mixing.

Now suppose that $C \in \mathcal{F}$ is an invariant set under $T_{\mathbf{z}}$. Since $T_{\mathbf{z}}^{-k}C = C$ for all $k \geqslant 0$, by setting $A = B = C$ we have that

$$P_p(C) = \lim_{k \to \infty} P_p(C \cap T_{\mathbf{z}}^{-k}C) = P_p(C)P_p(C) \tag{5.3}$$

and hence either $P_p(C) = 0$ or $P_p(C) = 1$.

■

Let $N(\omega)$ denote the number of distinct unbounded clusters present in a particular configuration $\omega \in \Omega'$ and let A_q denote the set $\{N(\omega) = q\}$. The next two lemmas show that A_q is measurable and that $N(\omega)$ is almost surely constant.

LEMMA 5.5: Every A_q, $q \in \{0, 1, \dots, \infty\}$, is $\mathcal{F}$-measurable.

Proof: First suppose that $q < \infty$ and that $\omega \in \Omega'$ is a configuration for which $N(\omega) = q$. Then there exists a random constant $L \geqslant 0$ such that all unbounded clusters intersect both $B(L)$ and $A(K)$ for all $K \geqslant L$.

For $n \geqslant 1$ and $K \geqslant L \geqslant 0$, let $E_{n,q}^{L,K}$ denote the event $\{C'_n \cap B(K+1)$ contains at least q distinct clusters intersecting both $B(L)$ and $A(K)\}$ and observe that we may write

$$A_q = \bigcap_{n\geqslant 1} \bigcup_{L\geqslant 0} \bigcap_{K\geqslant L} E_{n,q}^{L,K} \Big\backslash \bigcap_{n\geqslant 1} \bigcup_{L\geqslant 0} \bigcap_{K\geqslant L} E_{n,q+1}^{L,K}. \tag{5.4}$$

Each $E_{n,q}^{L,K}$ depends only on the states of finitely many squares and hence is a union of cylinder subsets of Ω', *i.e.* $E_{n,q}^{L,K} \in \mathcal{F}$. Clearly, therefore, each A_q, $q < \infty$, is $\mathcal{F}$-measurable; also A_∞ is $\mathcal{F}$-measurable since $A_\infty = \Omega' \setminus \bigcup_{q \geqslant 0} A_q$.

■

LEMMA 5.6: There exists $q \in \{0, 1, \ldots, \infty\}$ such that $N(\omega) = q$ for P_p-almost all $\omega \in \Omega'$.

Proof: Choose $\mathbf{z} \in \mathbb{Z}^2 \backslash \{(0,0)\}$. Since the total number of clusters in any particular configuration is not affected by translation, the event A_q is invariant under $T_{\mathbf{z}}$. By Theorem 5.4, P_p is ergodic with respect to $T_{\mathbf{z}}$, and hence either $P_p(A_q) = 0$ or $P_p(A_q) = 1$.

■

Proof of Theorem 5.1:

For $K \geqslant 0$, let $D(K)$ denote the square annulus $B(M^{K+1}) \backslash \text{int } B(M^K)$ and let $H(K)$ denote the event $\{C'_\infty \cap D(K)$ contains a cluster enclosing the origin$\}$. By the usual pasting results (see page 14), it is easy to see that $P_p(H(0)) = \varepsilon = \varepsilon(p) > 0$ whenever $p \geqslant p_c$. On rescaling by a factor of M, observe that the annulus $D(K)$ maps to $D(K+1)$ and that the probability of $H(K+1)$ is precisely the probability of $H(K)$ conditional on full retention at level 1. We deduce that $P_p(H(K+1)) \geqslant P_p(H(K)) \geqslant \varepsilon$ for all $K \geqslant 0$. By the Borel–Cantelli Lemma, therefore, we see that $H(K)$ occurs for infinitely many K with probability 1.

Let E be an unbounded cluster of C_∞; then there exists $L = L(E) \geqslant 0$ such that E intersects $B(M^L)$. Since E has infinite diameter, $E \cap D(K)$ must contain a connected component intersecting both internal and external boundaries of the annulus $D(K)$ for every $K \geqslant L$. Therefore if $H(K)$ occurs for some $K \geqslant 0$, then there exists at most one distinct unbounded cluster intersecting $B(M^K)$ (any such unbounded clusters being necessarily joined together by a component of $C'_\infty \cap D(K)$, as in Figure 5.1). Since $H(K)$ occurs for infinitely many K with probability 1, we deduce that C'_∞ contains at most one unbounded cluster, almost surely; from Lemma 5.6 and the result of Chayes *et al.* [8] we conclude that there exists exactly one unbounded cluster, almost surely.

■

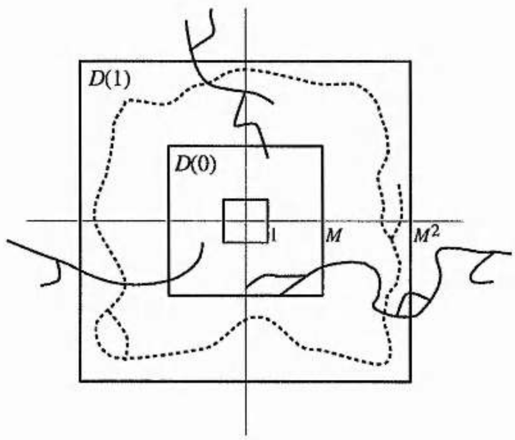


Figure 5.1: At most one unbounded cluster intersects $D(1)$

5.1.2 Nature of the unbounded cluster

In order to prove the uniqueness of the infinite cluster in the case of site percolation on the d-dimensional lattice, Burton and Keane [5] introduced the concept of encounter points: A point on the lattice is termed an encounter point if its removal breaks an infinite open cluster into three distinct infinite clusters. They showed that almost surely, encounter points occur with zero density throughout the lattice, and from this deduced that the expected number of infinite clusters is finite. An intersection argument then shows that the infinite cluster is almost surely unique. Meester [42] slightly simplified their argument by removing the requirement for ergodicity.

In this section, we shall introduce a modified version of encounter points applicable to fractal percolation, namely encounter squares, and prove that these encounter squares also occur with zero density, almost surely. This result has important consequences for an understanding of the connectivity of the unbounded cluster; it means that for all $p \geqslant p_c$, the unbounded cluster is typically very highly connected. More precisely, it is not generally disconnected by the removal of a bounded region and thus must possess relatively many interconnections between distant parts of the cluster. This is to be contrasted with the so-called incipient infinite cluster that is conjectured to arise at the critical value in the case of discrete percolation; this incipient cluster consists mainly of elongated, dendritic paths with relatively few interconnections.

Let S be a square of the form $\mathbf{z} + [0,1]^2$, $\mathbf{z} \in \mathbb{Z}^2$. We say that S is an *encounter square* for C'_∞ if S intersects an unbounded cluster F of C'_∞ and $F \backslash \operatorname{int} S$ consists of at least three distinct unbounded clusters.

LEMMA 5.7: The event $\{[0,1]^2$ is an encounter square$\}$ is $\mathcal{F}$-measurable.

Proof: For $n \geqslant 1$ and $K \geqslant 1$, let E_n^K denote the event $\{C'_n \cap B(K)$ contains a cluster F such that $F \backslash (0,1)^2$ contains at least three distinct clusters intersecting both $\partial[0,1]^2$ and $\partial B(K)\}$ and observe that we may write

$$\{[0,1]^2 \text{ is an encounter square}\} = \bigcap_{n \geqslant 1} \bigcap_{K \geqslant 1} E_n^K. \tag{5.5}$$

The result follows since each E_n^K depends only on the states of finitely many squares and hence is clearly $\mathcal{F}$-measurable.

■

THEOREM 5.8: For all $0 \leqslant p \leqslant 1$, we have

$$P_p\big([0,1]^2 \text{ is an encounter square}\big) = 0.$$

In order to prove Theorem 5.8, we shall require some geometrical results. We show that if we assume encounter squares occur with positive density throughout the plane, then the number of connected components of C'_∞ that intersect both the internal and external boundaries of $A(K)$ scales as K^2 as $K \to \infty$. This in turn would imply that the expected number of crossings of the unit square is unbounded; however, Meester [41] has shown that this number is in fact finite, and so our assumption was false.

Given a particular realisation of C'_∞ and $K \geqslant 1$, let $\#N(K)$ denote the number of encounter squares contained in $B(K)$. We partition $C'_\infty \cap A(K)$ into a set of connected components, and say that a component is an *exit* if it has non-empty intersection with both $\partial B(K)$ and $\partial B(K+1)$; let $\#X(K)$ denote the number of exits in $C'_\infty \cap A(K)$. It is readily seen that $\#N(K)$ and $\#X(K)$ are both $\mathcal{F}$-measurable random variables; the following purely geometrical result establishes an inequality between them.

PROPOSITION 5.9: For all $K \geqslant 1$, if $\#N(K) > 0$ then $\#X(K) \geqslant \#N(K) + 2$.
Proof: Fix $K \geqslant 1$, let F be a fixed unbounded cluster of C'_∞ and let Y be the set of exits of C^K_∞ contained in F. Suppose that $S \subseteq B(K)$ is an encounter square for F. Then the removal of S from F defines a partition $P = \{P_1, P_2, P_3\}$ of Y, with each P_i non-empty. If S' is another encounter square for F with corresponding partition $Q = \{Q_1, Q_2, Q_3\}$, it is easy to see that the indices for P and Q may be chosen so that $P_1 \subseteq Q_2 \cup Q_3$; we say then that P and Q are *compatible*. A collection $\mathcal{P}$ of partitions is termed compatible if each pair $P, Q \in \mathcal{P}$ is compatible. We now require the following sub-lemma.

LEMMA 5.10: Suppose that $\mathcal{P}$ is a compatible collection of partitions of Y; then $\#\mathcal{P} \leqslant \#Y - 2$.
Proof: See Burton and Keane [5], p. 504. ∎

Continuing with the proof of Proposition 5.9, we see that by Lemma 5.10 the number of exits contained in F is at least two more than the number of encounter squares for F in $B(K)$. Summing over each such unbounded cluster F of C'_∞ containing at least one encounter square within $B(K)$ completes the proof.

∎

Let $a, b \in \mathbb{Z}$, let $S = S(a, b)$ be the unit square $[a, a+1] \times [b, b+1]$ and let $C_S = C_\infty[(a, b)]$. We define an *H-crossing* of S to be a connected component of C_S that intersects both left and right sides of S, and let $N_H(S)$ be the number of distinct H-crossings of S. Meester [41] showed that $\mathbf{E}(N_H(S)) \leqslant \theta(p)^{-1}$.

Now for $a, b \in \mathbb{Z}$, let $R = R(a,b)$ denote the rectangle $[a, a+3] \times [b, b+1]$ and let $C_R = \bigcup_{i=0}^{2} C_\infty[(a+i, b)]$, that is, fractal percolation in R obtained by placing three independent copies of C_∞ side by side. An *H-crossing* of R is a connected component of C_R intersecting both left and right sides of R, a *V-crossing* intersects both top and bottom sides of R and a *T-crossing* intersects both $[a, a+3] \times \{b+1\}$ and $[a+1, a+2] \times \{b\}$; these crossings are illustrated in Figure 5.2. Let $N_H(R), N_V(R), N_T(R)$ denote the number of H-, V- and T-crossings of R respectively.

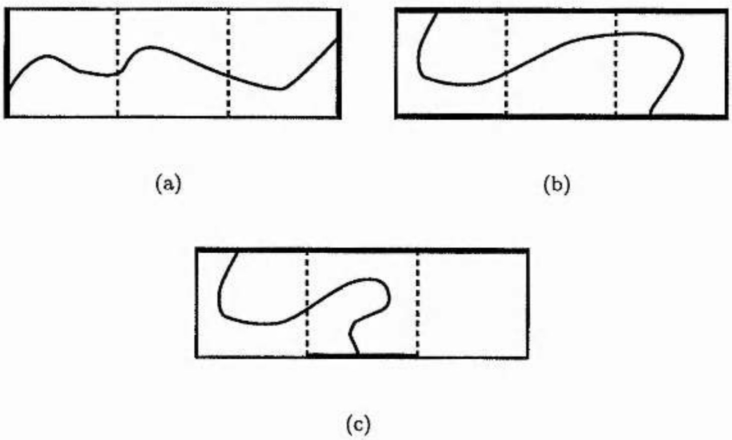


Figure 5.2: H-, V- and T-crossings of the rectangle R

LEMMA 5.11: Every exit contains either a T-crossing of some 3×1 rectangle R or an H-crossing of some unit square S (or a rotation of these events).

Proof: Fix $K \geqslant 1$ and let X be an exit in $C'_\infty \cap A(K)$. Without loss of generality, we may assume that X intersects $\partial B(K)$ at a point of $[a, a+1] \times \{K\}$, where $a \in \{-K, \dots, K-1\}$.

First suppose that $-K < a < K-1$. Let $R = [a-1, a+2] \times [K, K+1]$; then either $X \subseteq R$, in which case X is a T-crossing of R, or $X \not\subseteq R$, in which case X contains an H-crossing of either $[a-1, a] \times [K, K+1]$ or $[a+1, a+2] \times [K, K+1]$.

Next suppose that $a = K-1$ (or, equivalently, $a = -K$). In addition to the options above, there is also the possibility that X may 'go around the corner' of $\partial B(K)$. In this case, let $R = [K, K+1] \times [K-1, K+2]$ and

$S = [K, K+1] \times [K-1, K]$. Then it is easy to see that X must either contain a rotated T-crossing of R or a V-crossing (rotated H-crossing) of S.

■

LEMMA 5.12: The expected number of T-crossings of a 3×1 rectangle R is finite.

Proof: Let $\theta^{(3)}(p) = P_p(\exists\ H\text{-crossing of } R)$; by Lemma 1.8, we have that $\theta^{(3)}(p) > 0$ whenever $p \geqslant p_c$. Then R admits a *vacant* V-crossing (that is, $R \backslash C_R$ contains a path-connected component intersecting both top and bottom of R) with probability $1 - \theta^{(3)}(p)$, and following the arguments in the proof of Theorem 2.4 of Meester [41] we obtain

$$P_p(R \text{ admits at least } k \text{ vacant } V\text{-crossings}) \leqslant \left(1 - \theta^{(3)}(p)\right)^k \tag{5.6}$$

and hence $P_p(N_V(R) \geqslant k) \leqslant \left(1 - \theta^{(3)}(p)\right)^{k-1}$ for all $k \geqslant 1$. Therefore

$$\mathbf{E}\big(N_V(R)\big) = \sum_{k=1}^{\infty} P_p\big(N_V(R) \geqslant k\big) \leqslant \sum_{k=1}^{\infty}\left(1 - \theta^{(3)}(p)\right)^{k-1} = \theta^{(3)}(p)^{-1}. \tag{5.7}$$

Since $N_T(R) \leqslant N_V(R)$ trivially, we also have

$$\mathbf{E}\big(N_T(R)\big) \leqslant \theta^{(3)}(p)^{-1}. \tag{5.8}$$

■

Proof of Theorem 5.8:

Suppose that $P_p([0,1]^2 \text{ is an encounter square}) = \varepsilon > 0$. Since P_p is invariant under translation by $\mathbf{z} \in \mathbb{Z}^2$, every square $\mathbf{z} + [0,1]^2$ has probability ε of being an encounter square and hence

$$\mathbf{E}\big(\#N(K)\big) = 4K^2\varepsilon \tag{5.9}$$

for all $K \geqslant 1$.

For $K \geqslant 1$, define $N_H(K) = \sum_S (N_H(S) + N_V(S))$ and $N_T(K) = \sum_R N_T(R)$, where the sums are taken over all unit squares S and all 3×1 or 1×3 rectangles R with centres lying in $A(K)$. By Lemma 5.11, therefore, we have

$$\begin{aligned} N_H(K) + N_T(K) &\geqslant \#X(K) \\ &\geqslant \#N(K) + 2 \end{aligned} \tag{5.10}$$

if $\#N(K) > 0$, by Proposition 5.9. Taking expectations, we see that either $\mathbf{E}(N_H(K)) \geqslant 2K^2\varepsilon$ or $\mathbf{E}(N_T(K)) \geqslant 2K^2\varepsilon$ for infinitely many K; we shall assume (without loss of generality, since every H-crossing may be reformulated as a T-crossing) that

$$\mathbf{E}\big(N_T(K)\big) \geqslant 2K^2\varepsilon \tag{5.11}$$

for infinitely many K.

However, since there are only $8(K+1)$ positions for 3×1 or 1×3 rectangles around the perimeter of $B(K)$, we have

$$\mathbf{E}\big(N_T(K)\big) = \sum_R \mathbf{E}\big(N_T(R)\big) \leqslant 8(K+1)\theta^{(3)}(p)^{-1} \tag{5.12}$$

for all $K \geqslant 1$, by (5.8). Taking K sufficiently large so that

$$8(K+1)\theta^{(3)}(p)^{-1} < 2K^2\varepsilon \tag{5.13}$$

thus contradicts (5.11). We conclude that with probability 1, encounter squares occur with zero density throughout the plane.

■

5.2 Open questions

Finally we present a miscellany of open questions concerning the standard fractal percolation process and other closely related models. Some are problems that the author has encountered during the course of this research; others offer whole new avenues of investigation.

Random Sierpiński carpet

Here, we work in the unit square $[0,1]^2$ and let $M = 3$ and $0 \leqslant p \leqslant 1$. We mimic the construction of the random Cantor set, but with the additional rule that at every level we always discard the centre square of each group of nine squares (the other squares being retained independently at random with probability p, as before). The resulting random set is thus the intersection of a random Cantor set with the 'Sierpiński carpet' (see Mandelbrot [39], page 144). Dekking and Meester [17] investigated this set in detail, and proved the existence of at least six distinct phases of behaviour as p varies.

We define the concept of percolation in the random Sierpiński carpet, the percolation function $\theta'(p)$ and the critical probability p_c' in exactly the same way as before. By comparison with the usual random Cantor set C_∞, it is easy to see that $\theta'(p) \leqslant \theta(p)$ and hence $p_c' \geqslant p_c$; it is now a long-standing open problem to prove that $p_c' > p_c$. This problem should be contrasted with the analogous result of Aizenman and Grimmett [1] in the case of discrete percolation, where a periodically repeated enhancement of the lattice is shown to result in a reduction of the critical probability.

More generally, suppose that instead of retaining every square with the same probability p, we allow the retention probability to vary with location (but not with level). That is, for $1 \leqslant i \leqslant 9$ we choose p_i between 0 and 1 and retain S_i with probability p_i, where the S_i are the level-1 squares defined by Figure 4.4. This same pattern of probabilities is then repeated throughout the entire construction. The percolation function $\theta\colon [0,1]^9 \to [0,1]$ is defined by

$$\theta(\mathbf{p}) = P_{\mathbf{p}}(\text{percolation in } C_\infty),$$

where $\mathbf{p} = (p_1, \ldots, p_9)$. The phase transition to percolation is then represented by a surface $\partial\mathbf{P}$ in phase-space, where

$$\mathbf{P} = \{\mathbf{p}\colon \theta(\mathbf{p}) = 0\} \subseteq [0,1]^9.$$

What does this surface look like? Clearly $\mathbf{P}$ is convex; establishing that $\partial\mathbf{P}$ is nowhere parallel to the p_5-axis would be sufficient to prove that $p_c' > p_c$ (see Figure 5.3).

Continuity of the percolation function

In the case of discrete percolation, it is known that the percolation function is continuous everywhere (except possibly at the critical point in $d \geqslant 3$ dimensions). In fractal percolation, Chayes *et al.* [8] and Meester [41] showed the phase transition to be discontinuous, *i.e.* $\theta(p_c) > 0$, at least in two dimensions. It is not however known whether or not $\theta(p)$ is a continuous function above the critical point p_c; right-continuity of the percolation function is easy to establish, since $\theta(p)$ may be written as a decreasing limit of continuous increasing functions, but left-continuity has not been proved for $p_c < p < 1$.

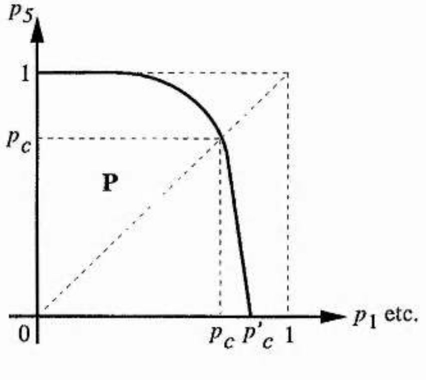


Figure 5.3: The region **P** in phase-space

Monotonicity of $p_c(M)$

There is some experimental evidence to suggest that the critical probability is monotonic decreasing in M, *i.e.* $p_c(M') \leqslant p_c(M)$ whenever $M' \geqslant M$. However, this inequality has only been proven for certain subsequences of M: By comparing the level-n set of fractal percolation with subdivision index M^2 to the level-$2n$ set of fractal percolation with subdivision index M, it is easy to see that

$$P_p(\text{percolation in } C_n[M^2]) \geqslant P_p(\text{percolation in } C_{2n}[M])$$

for all $n \geqslant 1$, and hence that $p_c(M^2) \leqslant p_c(M)$.

Self-affine fractal percolation

In this variation, instead of dividing $[0,1]^2$ into a mesh of squares, we choose two subdivision indices $M, N \geqslant 2$ and divide the unit square into a $M \times N$ mesh of rectangles in the natural way; each rectangle is then retained independently with probability p. Iterating this process, the limit set is statistically self-affine, *i.e.* small rectangles are statistical copies of larger rectangles, but scaled differently in different directions. Gatzouras and Lalley [27] considered the fractal dimension of this set, showing that the almost sure Hausdorff and box-counting dimensions are not in general equal, but proving that they are equal in an alternate model in which the rectangles are retained not independently but according to some fixed distribution.

Denoting the critical probability for percolation by $p_c(M, N)$, it is not hard to see that $p_c(M,N) = p_c(N,M)$, since any horizontal crossing of the squares includes a vertical crossing of some smaller rectangle, almost surely. We would like to prove inequalities of the form $p_c(M,M) \geqslant p_c(M,N) \geqslant p_c(N,N)$ for $M \leqslant N$, and $p_c(M,N) \geqslant p_c(M,N')$ for $N \leqslant N'$.

Dimension of crossings

In Chapter 2, we established almost sure upper and lower bounds on the minimal box-counting dimension of crossings in C_∞ in terms of p; these bounds could certainly be improved upon. There are also several other important questions unanswered: Does there exist an almost sure constant $\lambda_B = \lambda_B(p)$ such that, conditional on percolation occurring, the minimal box-counting dimension of all crossings is exactly λ_B, *i.e.*

$$P_p\big(\inf\{\dim_B(\Gamma)\colon \Gamma \text{ is a crossing}\} = \lambda_B\big) = 1 \quad ?$$

If so, is this infimum attained, and is λ_B continuous in p? We may ask the same questions for the Hausdorff dimension, defining λ_H to be the almost sure minimal Hausdorff dimensions of crossings; if λ_B and λ_H both exist, are they equal?

Percolation on random gaskets

Instead of working in the unit square, it is possible to define percolation processes on other objects, notably the Sierpiński gasket-type fractals. In the case of the triangular gasket, where we retain triangles independently at random with probability p, we find that the geometry of the gasket prevents percolation occurring, almost surely, for every value of $p < 1$. To see this, consider the level-n pre-fractal set, where $n \geqslant 1$. If two level-n triangles intersect, they do so only at a point, so any level-n crossing must pass through these level-n vertices; but with probability 1, all such vertices are eventually removed, so the limit set does not contain any crossings.

The process of fractal percolation is non-trivial in the case of other gaskets that are not post-critically finite. The *octogasket* is obtained by repeated substitution of a pattern of eight octagons intersecting along edges to form a ring, as shown in Figure 5.4. We define percolation to occur in the random octogasket if the limit set contains a complete circuit around the ring; since there is more

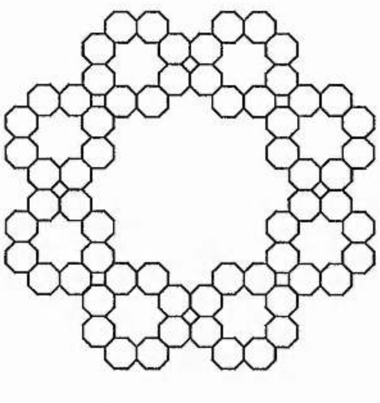


Figure 5.4: Construction of the octogasket

than one way to cross any edge, we can prove that the critical probability is strictly less than 1. Which other properties of fractal percolation have analogues in the random $4m$-gaskets, $m \geqslant 2$?

Percolation in higher dimensions

As already mentioned in the text, the higher-dimensional versions of fractal percolation are not well understood and there are several outstanding open problems. Defining arc-percolation to occur if the limit set contains a continuous map between opposite faces of the cubes, can we prove that arc-percolation is a measurable event, and if so, is it equivalent to the usual definition of percolation (up to sets of measure zero)? In three dimensions, letting p_c, p_a, p_s and p_d denote the critical probabilities for percolation, arc-percolation, sheet percolation and disc percolation respectively, we conjecture that $p_c = p_a < p_s = p_d$. In higher dimensions, we conjecture that $p_1 < p_2 < \ldots < p_{d-1} < 1$, where p_k is the critical probability for k-ball percolation in d dimensions (see Section 3.2.1 for definitions).

By analogy with the calculations of Meester [41] in two dimensions, it is easy to prove that the expected number of disjoint sheet crossings in fractal percolation in the cube is finite, since any two retained sheets are separated by a vacant sheet. Is the expected number of disjoint paths crossing the cube finite?

The pasting results cease to be so useful in three and higher dimensions, since we can have two or more paths crossing the unit cube that have no points in common. Some alternative intersection result for percolating paths in higher dimensions, for example a lower bound on the probability that two nearby paths are in fact joined together, would therefore be very helpful for extending the theory. In particular, we should then be able to modify the methods of Chayes *et al.* [8] and Burton and Keane [5] to prove the existence of an unbounded cluster in the tiling model in higher dimensions.

Resistance of fractal percolation

The discrete bond percolation process readily adapts to model a random electrical network, by giving each selected edge unit resistance and each vacant edge infinite resistance; the effective resistance of a large block of this random material can then be studied (see Section 10.9 of Grimmett [29]). Barlow and Bass [3] investigated the resistance of the Sierpiński carpet, proving that the effective resistance of the level-n pre-fractal set scales approximately as ρ^n as $n \to \infty$ for some $\rho \approx 1.2515$.

Crawford [14] set up a correspondence between the level-n sets of the fractal percolation process and a random resistor network, which is then used to study the hydraulic properties of a sample of soil modelled by a random Cantor set. In a similar vein, we could introduce capacity constraints on the level-n squares in fractal percolation to form a random capacitated network, and try to find the maximum permissible flow across the unit square when percolation occurs.

Dimension of cut-sets

Suppose that C_∞ is a realisation of the fractal percolation process in which percolation occurs. A set $T \subseteq [0,1]^2$ is called a *cut-set* if percolation does not occur in $T \backslash C_\infty$, *i.e.* every crossing of C_∞ uses at least one point of T. Does there exist an almost sure value χ_B (respectively, χ_H) for the minimal box-counting dimension (respectively, Hausdorff dimension) of all cut-sets T? How is χ_B (χ_H) related to the dimension of C_∞, or to the minimal dimension λ_B (λ_H) of crossings; if χ_B and χ_H both exist, are they equal? Returning to the capacitated model suggested above, does there exist an analogue of the 'max-flow min-cut' theorem for networks?

Mandelbrot aerogels

Aerogels are porous materials with pore sizes that exhibit scale invariance over a wide range of length scales. Machta [35] introduced a variant of the fractal percolation process known as the Mandelbrot aerogel; most of the important results to date for this model were established by Chayes, Chayes and Machta [10].

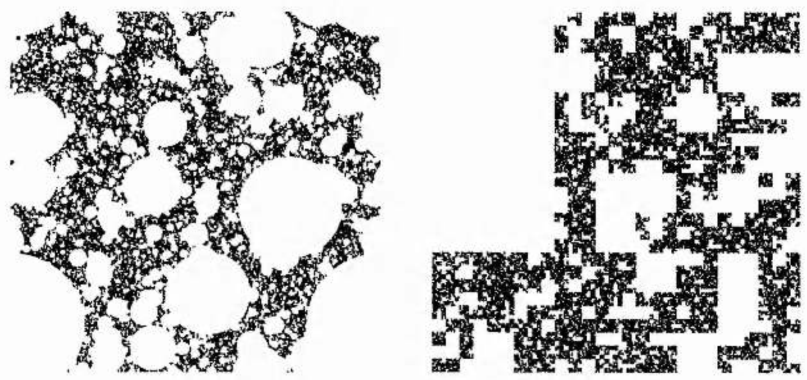
Briefly, the (two-dimensional) model is constructed as follows. Let $M \geqslant 2$ and let $0 \leqslant q, p \leqslant 1$; we generate a sequence $\{\tilde{C}_n\}_{n\geqslant 1}$ of subsets of the unit square as follows. Each $\tilde{C}_n$ is obtained by first running through the construction of the random Cantor set as far as level $(n-1)$ using the parameters M and q, obtaining a set C_{n-1}, and then performing one further iteration on C_{n-1}, but this time using the retention probability p. Unfortunately, there is no natural way to define a limiting set for this process, since the sets $\tilde{C}_n$ are not in general nested. Consequently, most of the results on this model are probabilistic statements that hold uniformly in n.

The percolation probability is defined by $\vartheta(q,p) = \liminf_{n\to\infty} \vartheta_n(q,p)$, where $\vartheta_n(q,p) = P_{q,p}(\text{percolation in } \tilde{C}_n)$. In the language of the model, we say that the system is in the *sol phase* if $\vartheta(q,p) = 0$, and in the *gel phase* if $\vartheta(q,p) > 0$. Open problems of interest are to establish the existence of the limit $\lim_{n\to\infty} \vartheta_n(q,p)$ for all values of q and p, and to determine the exact nature of the transition and the boundary in (q,p) phase-space between the sol and the gel phases. See Chayes [11] for a discussion of these and other issues.

Zähle's random cut-out set

The random cut-out set was introduced by Mandelbrot [36] and generalised and studied in depth by Zähle [60]; see also Falconer [22] for an account. In two dimensions, the simplest example of a random cut-out set is obtained by removing from the unit square a sequence of discs with decreasing radii $r_1 > r_2 > \ldots$ and independently distributed centres; note that we allow the discs to overlap and identify opposite sides of the square. The resulting limit set is closely related (qualitatively, at least) to the random Cantor set, see Figure 5.5.

So far, most attention has been paid to the problem of finding the Hausdorff dimension of such a set. However, we may also look for percolation in the



(a) (b)

Figure 5.5: A random cut-out set and a random Cantor set

random cut-out model and investigate how the connectivity of the limit set changes as we vary the sequence $\{r_n\}_{n\geqslant 1}$: Suppose that $r_n \sim cn^{-1/2}$ as $n \to \infty$. Then it can be shown that the box-counting dimension of the limit set E satisfies $\overline{\dim}_B(E) \leqslant 2 - \pi c^2$, with equality holding with positive probability; thus there exists a critical value for c separating a percolating phase from a dust-like phase. What other phase transitions exist? When the dimension of the limit set lies between 1 and 2 and percolation occurs, does there exist an almost sure value for the minimal dimension of crossings? In higher dimensional versions of the random cut-out set, are the percolation and sheet percolation phases distinct? Many of the techniques devised for the fractal percolation process should have analogues for the random cut-out model.

Appendix A

Computer Programs

A.1 Program `find_g`

This Pascal program calculates the coefficients of the polynomial $g(p, \alpha)$, representing the probability that a crack is transmitted from top to bottom according to the model described in Section 4.2.

```
{Declare program}

program find_g;

{Declare variables}

type config=array[1..9] of integer;

var
   loop,i1,i2,i3,i4,i5,i6,i7,i8,i9:integer;
   powerp,poweralpha,power1alpha:integer;
   s:config;
   p:array[0..7] of real;
   coeffs:array[0..9,0..10,0..10] of integer;

{Function to examine a configuration for a top-bottom crack}

function crack(state:config):boolean;
var i:integer;
   open,horiz,vert,access:array[1..9] of boolean;

{open = type O; horiz = type O,H,B; vert = type O,V,B}
```

```
begin
   for i:=1 to 9 do
   begin
      open[i]:=(state[i]=0);
      vert[i]:=(state[i]=2) or open[i];
   end;

   open[5]:=(state[5]=3);
   vert[5]:=odd(state[5]);
   horiz[5]:=open[5] or (state[5]>=6);
{Test if a crack is transmitted to the squares of the middle row}
   access[4]:=(open[4] and (vert[1] or open[2]))
      or (open[1] and vert[4]);
   access[6]:=(open[6] and (vert[3] or open[2]))
      or (open[3] and vert[6]);
   access[5]:=(open[5] and (open[1] or vert[2] or open[3]
      or access[4] or access[6])) or (vert[5] and open[2])
      or (horiz[5] and open[4] and access[4]) or (horiz[5]
      and open[6] and access[6]);
   access[6]:=access[6] or (horiz[5] and access[5] and open[4]);
   access[4]:=access[4] or (horiz[5] and access[5] and open[6]);

   if not (access[4] or access[5] or access[6])
      then crack:=false
{Test if a crack is transmitted to the squares of the bottom row}
   else begin
      access[7]:=(vert[7] and access[4] and open[4])
         or (open[7] and access[5] and open[5]);
      access[8]:=(vert[8] and access[5] and open[5])
         or (open[8] and ((open[4] and access[4]) or (vert[5]
         and access[5]) or (open[6] and access[6])));
      access[9]:=(vert[9] and access[6] and open[6])
         or (open[9] and access[5] and open[5]);
```

```
    crack:=(access[7] or access[8] or access[9]);
  end;
end;
```

{*Run through all the configurations*}

```
begin
  for i1:=0 to 2 do
  for i2:=0 to 2 do
  for i3:=0 to 2 do
  for i4:=0 to 2 do
  for i5:=3 to 7 do
  for i6:=0 to 2 do
  for i7:=0 to 2 do
  for i8:=0 to 2 do
  for i9:=0 to 2 do
  begin
    s[1]:=i1; s[2]:=i2; s[3]:=i3;
    s[4]:=i4; s[5]:=i5; s[6]:=i6;
    s[7]:=i7; s[8]:=i8; s[9]:=i9;
    if crack(s) then begin
```

{*Calculate coefficients of p, alpha, 1-alpha*}

```
      powerp:=0;
      poweralpha:=0;
      power1alpha:=0;
      for loop:=1 to 9 do
      begin
        if (not (s[loop]=0)) and (not (s[loop]=3)) then
          powerp:=powerp+1;
        if s[loop]=1 then power1alpha:=power1alpha+1;
        if s[loop]=2 then poweralpha:=poweralpha+1;
      end;
      if s[5]=4 then power1alpha:=power1alpha+2;
      if (s[5]=5) or (s[5]=6) then
      begin
```

```
            poweralpha:=poweralpha+1;
            power1alpha:=power1alpha+1;
         end;
         if s[5]=7 then poweralpha:=poweralpha+2;
```

{*Write to table of coefficients*}

```
         coeffs[powerp,poweralpha,power1alpha]:=
            coeffs[powerp,poweralpha,power1alpha]+1;
      end;
   end;
```

{*Display table of coefficients*}

```
   for i1:=0 to 9 do
   begin
      i4:=9-i1;
      write('p^',i1:1,'*(1-p)^',i4:1,'*(');
      for i2:=0 to 10 do
      begin
         for i3:=0 to 10 do
         if (coeffs[i1,i2,i3]>0) then
         write(coeffs[i1,i2,i3]:4,'*a^',i2:1,'*(1-a)^',i3:1,'+');
      end;
      writeln(') + ');
   end;

end.
```

A.2 Program `clusters` (schematic)

This program was used to generate realisations of the pre-fractal sets C_n, and calculate the number, size and diameter of the connected components at each level n.

Start

Declare variables

Get input from from keyboard:

    Subdivision parameter `M`

    Number of levels `maxdepth`

    Retention probability `p`

    Random number seed `seed`

Initialise arrays `state` *and* `looked_at` *of size* `maxsize:=`$\sum_{n=1}^{\mathtt{maxdepth}}(\mathtt{M}^2)^n$

    and type `boolean`

Function `up_level(i)`*:*

    Given a level-n square `i`*, returns the level-*$(n-1)$ *square containing* `i`

Function `rand`*:*

    Pseudo-random number generator, adapted from [50]

Fill array `state`*:*

    For `i=1` *to* `maxsize`

    If `rand<p` *and* `state[up_level(i)]=true` *then* `state[i]:=true`

        else `state[i]:=false`

Fill array `looked_at` *with* `false`

Procedure `dump_image`*:*

    Create pixmap image file

Procedure `find_clusters`*:*

For `i=1` *to* `maxsize`

    If `state[i]=false` *then exit*

    Open file `newlist` *for writing*

    Write `i` *to* `newlist`

    Let `looked_at[i]:=true, size:=1`

    Let `xmin:=xmax:=`*x-coord. of* `i`, `ymin:=ymax:=`*y-coord. of* `i`

    Repeat

```
        Copy newlist to oldlist
        Empty newlist
        Repeat
            Read i from oldlist
            Find neighbours j of i such that state[j]=true
                and looked_at[j]=false
            Write j to newlist
            Let looked_at[j]:=true, let size:=size+1
            If j is outwith [xmin, xmax] × [ymin, ymax]
                then update xmin,xmax,ymin,ymax
        Until end of file oldlist
    Until newlist is empty
    Let diam:=max(xmax-xmin,ymax-ymin)
    Output size, diam

Tabulate results
Stop.
```

References

[1] M. Aizenman and G. R. Grimmett. Strict monotonicity of critical points in percolation and ferromagnetic models. *J. Stat. Phys.*, **63**:817–835, 1991.

[2] K. B. Athreya and P. E. Ney. *Branching Processes.* Springer–Verlag, Berlin, 1972.

[3] M. T. Barlow and R. F. Bass. On the resistance of the Sierpiński carpet. *Proc. R. Soc. Lond. A*, **431**:345–360, 1990.

[4] P. Billingsley. *Ergodic Theory and Information.* John Wiley & Sons, New York, 1965.

[5] R. M. Burton and M. Keane. Density and uniqueness in percolation. *Commun. Math. Phys.*, **121**:501–505, 1989.

[6] M. Campanino and L. Russo. An upper bound on the critical percolation probability for the three-dimensional cubic lattice. *Ann. Prob.*, **13**:478–491, 1985.

[7] J. T. Chayes and L. Chayes. The large-N limit of the threshold values in Mandelbrot's fractal percolation process. *J. Phys. A: Math. Gen.*, **22**:L501–L506, 1989.

[8] J. T. Chayes, L. Chayes, and R. Durrett. Connectivity properties of Mandelbrot's percolation process. *Prob. Th. Rel. Fields*, **77**:307–324, 1988.

[9] J. T. Chayes, L. Chayes, E. Grannan, and G. Swindle. Phase transitions in Mandelbrot's percolation process in three dimensions. *Prob. Th. Rel. Fields*, **90**:291–300, 1991.

[10] J. T. Chayes, L. Chayes, and J. Machta. Phase diagram and correlation length bounds for Mandelbrot aerogels. *J. Phys. A: Math. Gen.*, **26**:4249–4271, 1993.

[11] L. Chayes. Aspects of the fractal percolation process. *Fractal Geometry and Stochastics* (Eds. C. Bandt, S. Graf and M. Zähle), 113–143. Birkhäuser, Basel, 1995.

[12] L. Chayes. On the absence of directed fractal percolation. *J. Phys. A: Math. Gen.*, **28**:L295–L301, 1995.

[13] L. Chayes. On the length of the shortest crossing in the super-critical phase of Mandelbrot's percolation process. *Stoch. Proc. Applic.*, **61**:25–43, 1996.

[14] J. W. Crawford. The relationship between structure and the hydraulic conductivity of soil. *Eur. J. Soil Sci.*, **45**:493–502, 1994.

[15] F. M. Dekking. Quadtrees, Mandelbrot percolation and the modelling of random sets. *Fractals*, **3**:777–784, 1995.

[16] F. M. Dekking and G. R. Grimmett. Superbranching processes and projections of random Cantor sets. *Prob. Th. Rel. Fields*, **78**:335–355, 1988.

[17] F. M. Dekking and R. W. J. Meester. On the structure of Mandelbrot's percolation process and other random Cantor sets. *J. Stat. Phys.*, **58**:1109–1126, 1990.

[18] A. Dvoretsky, P. Erdős, and S. Kakutani. Double points of paths of Brownian motion in n-space. *Acta Sci. Math. Szeged*, **12**:75–81, 1950.

[19] A. Dvoretsky, P. Erdős, S. Kakutani, and S. J. Taylor. Triple points of Brownian motion in 3-space. *Proc. Camb. Phil. Soc.*, **53**:856–862, 1957.

[20] K. J. Falconer. Random fractals. *Math. Proc. Camb. Phil. Soc.*, **100**:559–582, 1986.

[21] K. J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications.* John Wiley & Sons, Chichester, 1990.

[22] K. J. Falconer. *Techniques in Fractal Geometry.* John Wiley & Sons, Chichester, 1997.

[23] K. J. Falconer and G. R. Grimmett. The critical point of fractal percolation in three and more dimensions. *J. Phys. A: Math. Gen.*, **24**:L491–L494, 1991.

[24] K. J. Falconer and G. R. Grimmett. On the geometry of random Cantor sets and fractal percolation. *J. Theor. Prob.*, **5**:465–485, 1992.

[25] C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre. Correlation inequalities on some partially ordered sets. *Commun. Math. Phys.*, **22**:89–103, 1971.

[26] E. E. Fournier d'Albe. *Two New Worlds: I. The Infra World; II. The Supra World.* Longmans Green, London, 1907.

[27] D. Gatzouras and S. P. Lalley. Statistically self-affine sets: Hausdorff and box dimensions. *J. Theor. Prob.*, **7**:437–468, 1994.

[28] S. Graf. Statistically self-similar fractals. *Prob. Th. Rel. Fields*, **74**:357–392, 1987.

[29] G. R. Grimmett. *Percolation.* Springer–Verlag, New York, 1989.

[30] T. E. Harris. A lower bound for the critical probability in a certain percolation process. *Proc. Camb. Phil. Soc.*, **56**:13–20, 1960.

[31] T. E. Harris. *The Theory of Branching Processes.* Springer–Verlag, Berlin, 1963.

[32] J. Hawkes. Trees generated by a simple branching process. *J. London Math. Soc. (2)*, **24**:373–384, 1981.

[33] F. Hoyle. On the fragmentation of gas clouds into galaxies and stars. *Astrophys. J.*, **118**:513–528, 1953.

[34] H. Kesten. *Percolation Theory for Mathematicians.* Birkhäuser, Boston, 1982.

[35] J. Machta. Phase transitions in fractal porous media. *Phys. Rev. Lett.*, **66**:169–172, 1991.

[36] B. B. Mandelbrot. Renewal sets and random cutouts. *Z. Warsch. Verw. Geb.*, **22**:145–157, 1972.

[37] B. B. Mandelbrot. Intermittent turbulence in self-similar cascades: Divergence of high moments and dimension of the carrier. *J. Fluid Mech.*, **62**:331–358, 1974.

[38] B. B. Mandelbrot. *Fractals: Form, Chance and Dimension.* W. H. Freeman, San Francisco, 1977.

[39] B. B. Mandelbrot. *The Fractal Geometry of Nature.* W. H. Freeman, San Francisco, 1982.

[40] R. D. Mauldin and S. C. Williams. Random recursive constructions: Asymptotic geometric and topological properties. *Trans. Amer. Math. Soc.*, **295**:325–346, 1986.

[41] R. W. J. Meester. Connectivity in fractal percolation. *J. Theor. Prob.*, **5**:775–789, 1992.

[42] R. W. J. Meester. Lecture notes in percolation. *University of Utrecht*, Preprint 980, 1996.

[43] M. V. Menshikov. Coincidence of critical points in percolation problems. *Soviet Math. Doklady*, **33**:856–859, 1986.

[44] C. M. Newman and L. S. Schulman. Infinite clusters in percolation models. *J. Stat. Phys.*, **26**:613–628, 1981.

[45] C. M. Newman and L. S. Schulman. Number and density of percolating clusters. *J. Phys. A: Math. Gen.*, **14**:1735–1743, 1981.

[46] E. A. Novikov and R. W. Stewart. Intermittency of turbulence and the spectrum of fluctuations of energy dissipation. *Isvestia Akad. Nauk SSR; Seria Geofiz.*, **3**:408–413, 1964.

[47] M. E. Orzechowski. On the phase transition to sheet percolation in random Cantor sets. *J. Stat. Phys.*, **82**:1081–1098, 1996.

[48] M. E. Orzechowski. A lower bound on the box dimension of crossings in fractal percolation. *Stoch. Proc. Applic.*, to appear.

[49] Y. Peres. Intersection-equivalence of Brownian paths and certain branching processes. *Commun. Math. Phys.*, **177**:417–434, 1996.

[50] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling. *Numerical Recipes: The Art of Scientific Computing.* Cambridge University Press, Cambridge, 1985.

[51] C. A. Rogers. *Hausdorff Measures.* Cambridge University Press, Cambridge, 1970.

[52] L. Russo. A note on percolation. *Z. Wahrsch. Verw. Gebiete*, **43**:39–48, 1978.

[53] R. T. Smythe and J. C. Wierman. *First passage percolation on the square lattice.* Springer–Verlag, Berlin, 1978.

[54] H. Su and Z. Q. Yan. Random fractal percolation analysis of ductile fracture. *Materials Lett.*, **16**:142–144, 1993.

[55] M. F. Sykes and J. W. Essam. Exact critical percolation proabilities for site and bond problems in two dimensions. *J. Math. Phys.*, **5**:1117–1127, 1964.

[56] J. van den Berg and A. Ermakov. An improvement of the Menshikov–Pelikh lower bound for the critical probability of site percolation on the square lattice. *Random Structures and Algorithms*, **8**:199–212, 1996.

[57] D. Williams. *Probability with Martingales.* Cambridge University Press, Cambridge, 1991.

[58] J. Wu and X. Liu. The existence of phase V in the Mandelbrot percolation process. *J. Stat. Phys.*, **80**:1405–1414, 1995.

[59] Y. Xu and W. Su. Modification and generalization of fractal percolation. *Progress Nat. Sci.*, **7**:148–154, 1997.

[60] U. Zähle. Random fractals generated by random cutouts. *Math. Nachr.*, **116**:27–52, 1984.

[61] S. A. Zuev. Threshold estimates for a square lattice. *Theor. Prob. Appl.*, **32**:551–553, 1988.